

POISSON REDUCTION AS A COISOTROPIC INTERSECTION

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ABSTRACT. We give a definition of coisotropic morphisms of shifted Poisson (i.e. \mathbb{P}_n) algebras which is a derived version of the classical notion of coisotropic submanifolds. Using this we prove that an intersection of coisotropic morphisms of shifted Poisson algebras carries a Poisson structure of shift one less. Using an interpretation of Hamiltonian spaces as coisotropic morphisms we show that the classical BRST complex computing derived Poisson reduction coincides with the complex computing coisotropic intersection. Moreover, this picture admits a quantum version using brace algebras and their modules: the quantum BRST complex is quasi-isomorphic to the complex computing tensor product of brace modules.

INTRODUCTION

Coisotropic intersections. A transversal intersection of two Lagrangian submanifolds in a symplectic manifold is a collection of points that one could count. In Lagrangian Floer theory one counts intersections of non-transverse Lagrangians using, roughly speaking, the Morse complex of the symplectic action functional: the critical locus of the symplectic action functional is exactly the intersection.

Pantev, Toën, Vaquié and Vezzosi [PTVV] gave a derived-geometric interpretation of this result. Namely, the derived critical locus of the symplectic action functional is the derived Lagrangian intersection which was shown to have a (-1) -shifted symplectic structure. The Floer complex can then be thought of as a quantization of the Lagrangian intersection. This is a generalization of a result of Behrend and Fantechi [BF10] who show that the cohomology of a derived Lagrangian intersection carries a (-1) -shifted Poisson (\mathbb{P}_0) structure. More generally, derived intersection of two Lagrangians in an n -shifted symplectic stack was shown to be $(n-1)$ -shifted symplectic.

Baranovsky and Ginzburg [BG09] generalized the Behrend–Fantechi result in a different direction. Namely, they have shown that the cohomology of a derived intersection of coisotropic submanifolds of a Poisson manifold carries a \mathbb{P}_0 -structure. It is thus natural to ask whether one can lift the Baranovsky–Ginzburg construction to the chain level.

Calaque, Pantev, Toën, Vaquié and Vezzosi [CPTVV] introduced n -shifted Poisson structures on derived stacks and derived coisotropic structures on morphisms of stacks. Let us recall their definitions in the affine setting. Let $X = \mathrm{Spec} A$ be an affine derived scheme. An n -shifted Poisson structure on X is the same as a \mathbb{P}_{n+1} -structure on A (this equivalence is a theorem of Melani [Mel14]). That is, A is a cdga together with a Poisson bracket of degree $-n$. For $Y = \mathrm{Spec} B$, another affine derived scheme, CPTVV define a coisotropic structure on a morphism $Y \rightarrow X$ to be the same as a \mathbb{P}_n -structure on B together with a morphism of \mathbb{P}_{n+1} -algebras $A \rightarrow \mathrm{End}_{\mathbb{P}_n}(B)$. On the right we have $\mathrm{End}_{\mathbb{P}_n}(B)$, the endomorphism algebra of B in the category of \mathbb{P}_n -algebras. A priori $\mathrm{End}_{\mathbb{P}_n}(B)$ is only an $\mathbb{E}_1 \otimes_{BV} \mathbb{P}_n$ -algebra

(here \otimes_{BV} is the Boardman–Vogt tensor product of operads), but a result announced by Rozenblyum (additivity of the Poisson operad) asserts that $\mathbb{E}_1 \otimes_{BV} \mathbb{P}_n \cong \mathbb{P}_{n+1}$.

This definition is expected to give rather easily a \mathbb{P}_n -structure on a coisotropic intersection. However, the additivity of the Poisson operad is not given by explicit formulas, so the explicit Poisson structure on the coisotropic intersection would be difficult to write down.

In this paper we develop coisotropic structures in the affine setting, i.e. for arbitrary commutative differential graded algebras. It is expected that the endomorphism algebra $\text{End}_{\mathbb{P}_n}(B)$ is closely related to the complex of $(n-1)$ -shifted polyvector fields

$$Z(B) = \widehat{\text{Sym}}(T_B[-n])$$

with the differential twisted by the Poisson structure on B .

This allows one to instead define coisotropic morphisms as \mathbb{P}_{n+1} -morphisms $A \rightarrow Z(B)$ (Definition 1.4). Note that the \mathbb{P}_{n+1} -structure on $Z(B)$ is very explicit: it is given by the Schouten bracket (i.e. by the commutator of derivations). Using this definition we prove the following theorem (Theorem 1.9).

Theorem. *Let A be a \mathbb{P}_{n+1} -algebra and $A \rightarrow B_1$, $A \rightarrow B_2$ two coisotropic morphisms. Then the derived intersection $B_1 \otimes_A^{\mathbb{L}} B_2$ carries a homotopy \mathbb{P}_n -structure. Moreover, the natural projection $B_1^{\text{op}} \otimes B_2 \rightarrow B_1 \otimes_A^{\mathbb{L}} B_2$ is a \mathbb{P}_n -morphism where B_1^{op} denotes the same cdga with the opposite Poisson bracket.*

The proof of this theorem uses ideas from Koszul duality. Since $\mathbb{P}_{n+1} \cong \mathbb{E}_1 \otimes_{BV} \mathbb{P}_n$, we expect the Koszul dual coalgebra of a \mathbb{P}_{n+1} -algebra to carry a compatible \mathbb{P}_n -structure; indeed, it is given by explicit formulas using the bar complex (Proposition 1.6). Similarly, we show that the Koszul dual to the A -module B_i carries a homotopy \mathbb{P}_n -structure given by the coisotropic structure. Finally, the derived tensor product $B_1 \otimes_A^{\mathbb{L}} B_2$ can be written as an underived cotensor product on the Koszul dual side.

Moment maps. We give an application of derived coisotropic intersection to Hamiltonian reduction.

Let us recall that given a symplectic manifold X with a G -action preserving the symplectic form, a moment map is a G -equivariant morphism $\mu: X \rightarrow \mathfrak{g}^*$ which is a Hamiltonian for the G -action. Hamiltonian reduction is defined to be

$$X//G = [\mu^{-1}(0)/G].$$

If 0 is a regular value for μ and the G -action on $\mu^{-1}(0)$ is free and proper, the quotient is a symplectic manifold as shown by Marsden and Weinstein. If one of these conditions fails, the quotient is only a stratified symplectic manifold which hints that it is a shadow of a derived symplectic structure.

Indeed, one can rewrite

$$X//G \cong [\text{pt}/G] \times_{[\mathfrak{g}^*/G]} [X/G].$$

Moreover, as shown in [Cal13] and [Saf13], Hamiltonian G -spaces are the same as Lagrangians in the 1-shifted symplectic stack $[\mathfrak{g}^*/G]$. Therefore, $X//G$ is a Lagrangian intersection and so carries a derived symplectic structure.

In this paper we show similar statements in the Poisson setting. Namely, if B is a \mathbb{P}_1 -algebra with $\mu: \text{Sym } \mathfrak{g} \rightarrow B$ a moment map for a \mathfrak{g} -action on B we show that the induced

morphism

$$C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g}) \rightarrow C^\bullet(\mathfrak{g}, B)$$

is coisotropic. Here $C^\bullet(\mathfrak{g}, -)$ is the Chevalley–Eilenberg cochain complex and $C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g})$ is the \mathbb{P}_2 -algebra of functions on the quotient $[\mathfrak{g}^*/G]$ with G formal.

The coisotropic intersection

$$C^\bullet(\mathfrak{g}, k) \otimes_{C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g})}^{\mathbb{L}} C^\bullet(\mathfrak{g}, B)$$

is thus a derived Poisson reduction which we show to be quasi-isomorphic to the classical BRST complex as defined by Kostant and Sternberg [KS87].

Quantization. We also develop quantum versions of our results. Namely, while quantum versions of \mathbb{P}_1 -algebras are dg algebras, quantum versions of \mathbb{P}_2 -algebras are \mathbb{E}_2 -algebras which we model by brace algebras. We introduce a notion of a brace module over a brace algebra which is a quantum version of a coisotropic morphism from a \mathbb{P}_2 -algebra. One way to think of it is as follows: the pair (brace algebra, brace module) is the same as an algebra over the Swiss-cheese operad introduced by Voronov [Vor98]. We prove the following quantum version of the coisotropic intersection theorem (Theorem 3.5).

Theorem. *Let A be a brace algebra, B_1 a left brace module and B_2 a right brace module over A . Then the derived tensor product $B_1 \otimes_A^{\mathbb{L}} B_2$ carries a natural dg algebra structure such that the projection $B_1^{\text{op}} \otimes B_2 \rightarrow B_1 \otimes_A^{\mathbb{L}} B_2$ is an algebra morphism.*

We apply this result to quantum moment maps. A quantization of the \mathbb{P}_2 -algebra $C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g})$ is the brace algebra $\text{CC}^\bullet(\text{U}\mathfrak{g}, \text{U}\mathfrak{g})$, the Hochschild cochain complex of the universal enveloping algebra $\text{U}\mathfrak{g}$. We show that a quantum moment map $\text{U}\mathfrak{g} \rightarrow B$ makes $\text{CC}^\bullet(\text{U}\mathfrak{g}, B)$ into a brace module over $\text{CC}^\bullet(\text{U}\mathfrak{g}, \text{U}\mathfrak{g})$. The tensor product

$$\text{CC}^\bullet(\text{U}\mathfrak{g}, k) \otimes_{\text{CC}^\bullet(\text{U}\mathfrak{g}, \text{U}\mathfrak{g})}^{\mathbb{L}} \text{CC}^\bullet(\text{U}\mathfrak{g}, B)$$

computing derived quantum Hamiltonian reduction is therefore a dga which is shown to be quasi-isomorphic to the quantum BRST complex.

This point of view on quantum Hamiltonian reduction allows one to generalize ordinary (i.e. \mathbb{E}_1) Hamiltonian reduction to \mathbb{E}_n -algebras which we discuss in Section 4.5.

Both classical and quantum constructions can be put on the same footing if one starts with a deformation quantization for which we use the language of Beilinson–Drinfeld algebras [CG15, Section 8.4]. We end the paper with some theorems that interpolate between classical coisotropic intersections and tensor products of brace modules.

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Notation. We work over a field k of characteristic zero. By a dga we mean a differential graded algebra over k not necessarily non-positively graded. For A a dga and M and N two modules we denote by $M \otimes_A^{\mathbb{L}} N$ the resolution given by the two-sided bar complex.

An (n, m) -shuffle $\sigma \in S_{n, m}$ is a permutation $\sigma \in S_{n+m}$ such that $\sigma(1) < \dots < \sigma(n)$ and $\sigma(n+1) < \dots < \sigma(n+m)$.

1. \mathbb{P}_n -ALGEBRAS

1.1. Polyvector fields. Let A be a cdga. We denote by $T_A = \text{Der}(A, A)$ the A -module of derivations which is a dg Lie algebra over k . We define the complex of $(n-1)$ -shifted polyvector fields to be $\text{Sym}(T_A[-n])$. We refer to the natural grading on the symmetric algebra as the *weight* grading.

The dual A -module of $\text{Sym}^k(T_A[-n])$ can be identified with $\text{Sym}^k(\Omega_A^1[n])$ for Ω_A^1 the module of Kähler differentials and we denote by \lrcorner the duality pairing. Given a polyvector $v \in \text{Sym}^k(T_A[-n])$ we define

$$(1) \quad v(a_1, \dots, a_k) = v \lrcorner (d_{\text{dR}} \otimes \dots \otimes d_{\text{dR}})(a_1 \otimes \dots \otimes a_k),$$

where d_{dR} is put in degree $-n$. The symmetry of v implies that

$$v(a_1, a_2, \dots, a_k) = (-1)^{|a_1||a_2|+n} v(a_2, a_1, \dots, a_k).$$

We define the Schouten bracket of $v \in \text{Sym}^k(T_A[-n])$ and $w \in \text{Sym}^l(T_A[-n])$ to be

$$\begin{aligned} [v, w](a_1, \dots, a_{k+l-1}) &= \sum_{\sigma \in S_{l, k-1}} \text{sgn}(\sigma)^n (-1)^{\epsilon_1} v(w(a_{\sigma(1)}, \dots, a_{\sigma(l)}), a_{\sigma(l+1)}, \dots, a_{\sigma(k+l-1)}) \\ &\quad - \sum_{\sigma \in S_{k, l-1}} \text{sgn}(\sigma)^n (-1)^{\epsilon_2} w(v(a_{\sigma(1)}, \dots, a_{\sigma(k)}), a_{\sigma(k+1)}, \dots, a_{\sigma(k+l-1)}), \end{aligned}$$

where $(-1)^\epsilon$ denotes the sign coming from the Koszul sign rule applied to the permutation σ of a_i and the signs ϵ_i are

$$\begin{aligned} \epsilon_1 &= (|w| + l)(k + 1)n + |v|n \\ \epsilon_2 &= (|v| - kn)(|w| - ln) + n(k + 1)(|w| + 1) + |v|n. \end{aligned}$$

The product of polyvector fields is defined to be

$$(v \cdot w)(a_1, \dots, a_{k+l}) = \sum_{\sigma \in S_{k, l}} \text{sgn}(\sigma)^n (-1)^{\epsilon_1} v(a_{\sigma(1)}, \dots, a_{\sigma(k)}) w(a_{\sigma(k+1)}, \dots, a_{\sigma(k+l)}),$$

where the sign is

$$\epsilon_1 = |w|kn + \sum_{i=1}^k |a_{\sigma(i)}|(nl + |w|).$$

1.2. Algebras. Let us begin with the basic object in this section which is a weak (and shifted) version of Poisson algebras.

Definition 1.1. A $\widehat{\mathbb{P}}_n$ -algebra is a cdga A together with an L_∞ -algebra structure of degree $1-n$ such that the L_∞ operations l_k are polyderivations with respect to the multiplication. More explicitly, l_k are multilinear operations of degree $1 - (k-1)n$ satisfying the following equations:

- (Symmetry).

$$l_k(a_1, \dots, a_i, a_{i+1}, \dots, a_k) = (-1)^{|a_i||a_{i+1}|+n} l_k(a_1, \dots, a_{i+1}, a_i, \dots, a_k).$$

- (Leibniz rule).

$$l_k(a_1, \dots, a_k a_{k+1}) = l_k(a_1, \dots, a_k) a_{k+1} + (-1)^{|a_k||a_{k+1}|} l_k(a_1, \dots, a_{k+1}) a_k.$$

- (Jacobi identity).

$$0 = \sum_{k=1}^m (-1)^{nk(m-k)} \sum_{\sigma \in S_{k,m-k}} \operatorname{sgn}(\sigma)^n (-1)^\epsilon l_{m-k+1}(l_k(a_{\sigma(1)}, \dots, a_{\sigma(k)}), a_{\sigma(k+1)}, \dots, a_{\sigma(m)}),$$

where ϵ is the sign coming from the Koszul sign rule.

Given a $\widehat{\mathbb{P}}_n$ -algebra A , the opposite algebra A^{op} is defined to be the same cdga together with operations $l_k^{\text{op}} = (-1)^{k+1} l_k$.

There is also a strict version of Poisson algebras as follows.

Definition 1.2. A \mathbb{P}_n -algebra is a $\widehat{\mathbb{P}}_n$ -algebra such that the operations l_m vanish for $m > 2$. In this case we denote the operation l_2 by $\{a, b\}$.

Here is an important example of a \mathbb{P}_{n+1} -algebra.

Proposition 1.3. Let A be a cdga. The product and Schouten bracket define a \mathbb{P}_{n+1} -structure on the complex of $(n-1)$ -shifted polyvector fields $\operatorname{Sym}(\mathcal{T}_A[-n])$.

A \mathbb{P}_n -structure on a cdga A is given by a bivector $\pi_A \in \operatorname{Sym}^2(\mathcal{T}_A[-n])[n+1]$, so that

$$\{a, b\} := \pi_A(a, b).$$

The Jacobi identity for the bracket then becomes

$$[\pi_A, \pi_A] = 0.$$

Given a \mathbb{P}_n -algebra A , we can naturally produce a \mathbb{P}_{n+1} -algebra $Z(A)$ called the *Poisson center* of A . As a graded commutative algebra, it is defined to be the completion of the algebra of $(n-1)$ -shifted polyvector fields

$$Z(A) = \widehat{\operatorname{Sym}}_A(\mathcal{T}_A[-n]).$$

The Lie bracket is given by the Schouten bracket. The differential has two components: the differential on the module of derivations and $[\pi_A, -]$.

Remark. The cohomology of $Z(A)$ is the Poisson cohomology of A . $Z(A)$ also coincides with the deformation complex of A as an algebra over the operad \mathbb{P}_n if one replaces \mathcal{T}_A by the tangent complex [CW13, Theorem 2].

We have a map

$$Z(A) \rightarrow A$$

of commutative dg algebras given by projecting to the weight zero part of polyvector fields.

1.3. Modules. Let A be a \mathbb{P}_{n+1} -algebra and M a cdga.

Definition 1.4. A *coisotropic structure* on a morphism of commutative dg algebras $f: A \rightarrow M$ is a \mathbb{P}_n -algebra structure on M and a lift

$$\begin{array}{ccc} A & \xrightarrow{\tilde{f}} & Z(M) \\ & \searrow f & \downarrow \\ & & M, \end{array}$$

where $\tilde{f}: A \rightarrow Z(M)$ is a map of \mathbb{P}_{n+1} -algebras.

Here is a way to unpack this definition. A coisotropic structure consists of maps

$$f_k: A \rightarrow \text{Sym}^k(\text{T}_M[-n])$$

for $k \geq 0$, where $f_0 = f$ is the original morphism. We define the maps

$$f_k: A \otimes M^{\otimes k} \rightarrow M[-nk]$$

by

$$f_k(a; m_1, \dots, m_k) := f_k(a)(m_1, \dots, m_k).$$

They satisfy the following equations:

- (Symmetry).
- (2) $f_k(a; m_1, \dots, m_i, m_{i+1}, \dots, m_k) = (-1)^{|m_i||m_{i+1}|+n} f_k(a; m_1, \dots, m_{i+1}, m_i, \dots, m_k)$
for every $a \in A$ and $m_i \in M$.
- (Derivation).
- (3) $f_k(a; m_1, \dots, m_k m_{k+1}) = f_k(a; m_1, \dots, m_k) m_{k+1} + (-1)^{|m_k||m_{k+1}|} f_k(a; m_1, \dots, m_{k+1}) m_k$
for every $a \in A$ and $m_i \in M$.
- (Compatibility with the differential).

$$(4) \quad df_k(a; m_1, \dots, m_k) =$$

$$\begin{aligned} & f_k(da; m_1, \dots, m_k) + \sum_{i=1}^k (-1)^{|a|+\sum_{j=1}^{i-1} |m_j|+nk} f_k(a; m_1, \dots, dm_i, \dots, m_k) \\ & - \sum_{i=1}^k (-1)^{n(|a|+i-1)+|m_i|} \sum_{j=i+1}^k |m_j| \{f_{k-1}(a; m_1, \dots, \hat{m}_i, \dots, m_k), m_i\} \\ & + \sum_{i < j} (-1)^{|m_i|} \sum_{l=1}^{i-1} |m_l| + |m_j| \sum_{l=1, l \neq i}^{j-1} |m_l| + n(i+j) + |a| f_{k-1}(a; \{m_i, m_j\}, m_1, \dots, \hat{m}_i, \dots, \hat{m}_j, \dots, m_k) \end{aligned}$$

for every $a \in A$ and $m_i \in M$.

- (Compatibility with the brackets).

For every $a_1, a_2 \in A$ and $m_i \in M$ we have

$$\begin{aligned} & f_k(\{a_1, a_2\}; m_1, \dots, m_k) \\ & = \sum_{i+j=k+1} \sum_{\sigma \in S_{j,i-1}} \text{sgn}(\sigma)^n (-1)^{\epsilon_1 + \epsilon_2} f_i(a_1; f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)}), m_{\sigma(j+1)}, \dots, m_{\sigma(k)}) \\ (5) \quad & - \sum_{i+j=k+1} \sum_{\sigma \in S_{j,i-1}} \text{sgn}(\sigma)^n (-1)^{\epsilon_1 + \epsilon_2} f_i(a_2; f_j(a_1; m_{\sigma(1)}, \dots, m_{\sigma(j)}), m_{\sigma(j+1)}, \dots, m_{\sigma(k)}), \end{aligned}$$

where the signs are

$$\begin{aligned} \epsilon_1 &= (|a_2| + j)(i + 1)n + |a_1|n \\ \epsilon_2 &= (|a_1| - jn)(|a_2| - in) + n(j + 1)(|a_2| + 1) + |a_1|n. \end{aligned}$$

- (Compatibility with the product).

For every $a_1, a_2 \in A$ and $m_i \in M$ we have

$$(6) \quad f_k(a_1 a_2; m_1, \dots, m_k) = \sum_{i+j=k} \sum_{\sigma \in S_{i,j}} \text{sgn}(\sigma)^n (-1)^{\epsilon + \epsilon_1} f_i(a_1; m_{\sigma(1)}, \dots, m_{\sigma(i)}) f_j(a_2; m_{\sigma(i+1)}, \dots, m_{\sigma(k)}),$$

where the sign is $\epsilon_1 = |a_2|ni + \sum_{l=1}^i |m_{\sigma(l)}|(nj + |a_2|)$.

Remark. Equation (5) for $k = 0$ reads as

$$f_0(\{a_1, a_2\}) = (-1)^{|a_1|n} f_1(a_1; f_0(a_2)) - (-1)^{n(|a_2|+1)+|a_1||a_2|} f_1(a_2; f_0(a_1)).$$

In particular, the kernel of f_0 is closed under the Poisson bracket and so $\text{Spec } M \rightarrow \text{Spec } A$ is a coisotropic submanifold in the usual sense. However, we do not know if all coisotropic submanifolds in the ordinary sense can be endowed with a coisotropic structure even in the case $n = 0$.

However, in [JS15, Section 3.6] Joyce and the author show that derived Lagrangians in derived symplectic schemes (in particular, ordinary smooth Lagrangians) can be *locally* endowed with a coisotropic structure.

1.4. Koszul duality. For a complex A we denote by $T_\bullet(A[1])$ the tensor coalgebra. As a complex,

$$T_\bullet(A[1]) \cong \bigoplus_{k=0}^{\infty} A^{\otimes k}[k].$$

We denote an element of $A^{\otimes k}$ by $[a_1 | \dots | a_k]$ for $a_i \in A$. The canonical element in $A^{\otimes 0}$ is denoted by $[\]$.

The coproduct is given by deconcatenation, i.e.

$$\Delta[a_1 | \dots | a_k] = \sum_{i=0}^k [a_1 | \dots | a_i] \otimes [a_{i+1} | \dots | a_k].$$

Let us denote by \wedge the concatenation product:

$$[a_1 | \dots | a_i] \wedge [a_{i+1} | \dots | a_k] = [a_1 | \dots | a_k].$$

Note that the deconcatenation coproduct and concatenation product do not form a bialgebra structure.

If A is a cdga, we can introduce the bar differential on $T_\bullet(A[1])$ and a commutative multiplication given by shuffles. That is,

$$\begin{aligned} d[a_1 | \dots | a_k] &= \sum_{i=1}^k (-1)^{\sum_{q=1}^{i-1} |a_q| + i - 1} [a_1 | \dots | da_i | \dots | a_k] \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\sum_{q=1}^i |a_q| + i} [a_1 | \dots | a_i a_{i+1} | \dots | a_k] \end{aligned}$$

and

$$[a_1 | \dots | a_k] \cdot [a_{k+1} | \dots | a_{k+m}] = \sum_{\sigma \in S_{k,m}} (-1)^\epsilon [a_{\sigma(1)} | \dots | a_{\sigma(k+m)}],$$

where the sign ϵ is determined by assigning degrees $|a_i| - 1$ to a_i . The element $1 \in T_\bullet(A[1])$ is the unit for the shuffle product. We refer the reader to [GJ90, Section 1] for a detailed explanations of all signs involved.

Now let A be a \mathbb{P}_{n+1} -algebra. Then we can define a Lie bracket on $T_\bullet(A[1])$ by

$$(7) \quad \{[a_1|\dots|a_k], [b_1|\dots|b_m]\} \\ = \sum_{i,j} (-1)^{\epsilon+|a_i|+n+1} ([a_1|\dots|a_{i-1}] \cdot [b_1|\dots|b_{j-1}]) \wedge \{a_i, b_j\} \wedge ([a_{i+1}|\dots|a_k] \cdot [b_{j+1}|\dots|b_m]).$$

The sign ϵ is determined by the following rule: an element b moving past $\{a, -\}$ produces a sign $(-1)^{(|b|+1)(|a|+n)}$. For instance,

$$\{[a], [b|c]\} = (-1)^{|a|+n+1} [\{a, b\}|c] + (-1)^{|b|(|a|+n)+1} [b|\{a, c\}].$$

Definition 1.5. A \mathbb{P}_n -bialgebra is a \mathbb{P}_n -algebra \tilde{A} together with a coassociative comultiplication $\tilde{A} \rightarrow \tilde{A} \otimes \tilde{A}$ which is a morphism of \mathbb{P}_n -algebras.

Proposition 1.6. *Thus constructed differential, multiplication, comultiplication and bracket define a \mathbb{P}_n -bialgebra structure on $T_\bullet(A[1])$.*

Proof. See [GJ90, Proposition 4.1] for the proof that $T_\bullet(A[1])$ is a commutative dg bialgebra. We just need to show that the bracket is compatible with the other operations.

Let us first show that the Lie bracket is compatible with the coproduct. We will omit some obvious signs arising from a permutation of a and b .

$$\begin{aligned} & \{\Delta[a_1|\dots|a_k], \Delta[b_1|\dots|b_m]\} \\ &= \sum_{i,j} \{[a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_k], [b_1|\dots|b_j] \otimes [b_{j+1}|\dots|b_m]\} \\ &= \sum_{i,j} (-1)^\epsilon \{[a_1|\dots|a_i], [b_1|\dots|b_j]\} \otimes ([a_{i+1}|\dots|a_k] \cdot [b_{j+1}|\dots|b_m]) \\ &+ \sum_{i,j} (-1)^\epsilon ([a_1|\dots|a_i] \cdot [b_1|\dots|b_j]) \otimes \{[a_{i+1}|\dots|a_k], [b_{j+1}|\dots|b_m]\} \\ &= \sum_{i,j,p,q} (-1)^\epsilon ([a_1|\dots|a_{p-1}] \cdot [b_1|\dots|b_{q-1}]) \wedge \{a_p, b_q\} \wedge ([a_{p+1}|\dots|a_i] \cdot [b_{q+1}|\dots|b_j]) \\ &\quad \otimes ([a_{i+1}|\dots|a_k] \cdot [b_{j+1}|\dots|b_m]) \\ &+ \sum_{i,j,p,q} (-1)^\epsilon ([a_1|\dots|a_i] \cdot [b_1|\dots|b_j]) \\ &\quad \otimes ([a_{i+1}|\dots|a_{p-1}] \cdot [b_{j+1}|\dots|b_{q-1}]) \wedge \{a_p, b_q\} \wedge ([a_{p+1}|\dots|a_k] \cdot [b_{q+1}|\dots|b_m]) \\ &= \Delta\{[a_1|\dots|a_k], [b_1|\dots|b_m]\}. \end{aligned}$$

In the last equality we have used that the tensor coalgebra with a shuffle product is a bialgebra.

The fact that the Lie bracket is symmetric is obvious from the graded commutativity of the shuffle product.

The Jacobi identity and the Leibniz rule are morphisms $f: T_\bullet(A[1])^{\otimes 3} \rightarrow T_\bullet(A[1])$ satisfying

$$\Delta_{T_\bullet(A[1])} \circ f = (f \otimes m + m \otimes f) \circ \Delta_{T_\bullet(A[1])^{\otimes 3}},$$

where $m: T_\bullet(A[1])^{\otimes 3} \rightarrow T_\bullet(A[1])$ is the multiplication map.

These are uniquely determined by the projections

$$T_\bullet(A[1])^{\otimes 3} \rightarrow T_\bullet(A[1]) \rightarrow A[1]$$

to cogenerators. Therefore, to check the relevant identities, we just need to see that the components landing in A are all zero.

- (Jacobi identity). The Lie bracket has a component in A only if both arguments are in A . Therefore, the Jacobi identity in $T_\bullet(A[1])$ reduces to the Jacobi identity in A itself.
- (Leibniz rule). The Leibniz rule

$$\{a, bc\} = \{a, b\}c + (-1)^{|b||c|}\{a, c\}b, \quad a, b, c \in T_\bullet(A[1])$$

has components in A only if either a or b are 1. In that case the Leibniz rule is tautologically true.

- (Compatibility with the differential). The compatibility relation

$$d\{a, b\} = (-1)^{n+1}\{da, b\} + (-1)^{|a|+n+1}\{a, db\}$$

has components in A if either both a and b are in A or one of them is in A and the other one is in $A^{\otimes 2}$. In the first case the compatibility of the bracket on $T_\bullet(A[1])$ with the differential reduces to the compatibility of the bracket on A with the differential. In the second case the A component of the equation is

$$(-1)^{|b_1|}\{a, b_1\}b_2 + (-1)^{|b_1|(|a|+n+1)}b_1\{a, b_2\} = (-1)^{|b_1|}\{a, b_1b_2\}.$$

After multiplying through by $(-1)^{|b_1|}$ we get the Leibniz rule for the bracket on A .

□

Remark. For A a \mathbb{P}_{n+1} -algebra the coalgebra $T_\bullet(A[1])^{\text{cop}}$ with the opposite coproduct is isomorphic to $T_\bullet(A^{\text{op}}[1])$ as a \mathbb{P}_n -bialgebra via

$$(8) \quad [a_1 | \dots | a_k] \mapsto (-1)^{k + \sum_{i < j} (|a_i|+1)(|a_j|+1)} [a_k | \dots | a_1].$$

1.5. Coisotropic intersection. Let us now describe a relative version of the previous statement. Let A be a \mathbb{P}_{n+1} -algebra and $f: A \rightarrow M$ a coisotropic morphism. We are going to define a \mathbb{P}_n -algebra structure on $T_\bullet(A[1]) \otimes M$, the one-sided bar complex of M . As before, we denote elements of $T_\bullet(A[1]) \otimes M$ by $[a_1 | \dots | a_k | m]$.

Recall that the bar differential is given by

$$\begin{aligned} d[a_1|...|a_k|m] &= \sum_{i=1}^k (-1)^{\sum_{q=1}^{i-1} |a_q| + i - 1} [a_1|...|da_i|...|a_k|m] \\ &\quad + (-1)^{\sum_{q=1}^k |a_q| + k} [a_1|...|a_k|dm] \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\sum_{q=1}^i |a_q| + i} [a_1|...|a_i a_{i+1}|...|a_k|m] \\ &\quad + (-1)^{\sum_{q=1}^k |a_q| + k} [a_1|...|a_k m]. \end{aligned}$$

One has an obvious coaction map making $T_\bullet(A[1]) \otimes M$ into a left dg $T_\bullet(A[1])$ -comodule. As a graded $T_\bullet(A[1])$ -comodule, $T_\bullet(A[1]) \otimes M$ is cofree.

Introduce a pointwise multiplication on $T_\bullet(A[1]) \otimes M$, where the multiplication on $T_\bullet(A[1])$ is given by shuffles as before and the multiplication on M is coming from its cdga structure. Note that

$$[a_1|...|a_k|m] = [a_1|...|a_k|1] \cdot [m].$$

The L_∞ operations we are about to introduce are multiderivations, so by the previous equation it is enough to specify them when the arguments are either in $T_\bullet(A[1])$ or in M . If all arguments are in $T_\bullet(A[1])$, we define the brackets as before. We let

$$(9) \quad l_{k+1}([a_1|...|a_p|1], [m_1], \dots, [m_k]) = (-1)^{(\sum_{q=1}^p |a_q| + p)(1 - nk)} [a_1|...|a_{p-1}|f_k(a_p; m_1, \dots, m_k)]$$

and

$$(10) \quad l_2([m_1], [m_2]) = [\{m_1, m_2\}],$$

where the Poisson bracket on the right is the bracket in M . All the other brackets are defined to be zero.

Definition 1.7. A left $\widehat{\mathbb{P}}_n$ -comodule \tilde{M} over a \mathbb{P}_n -bialgebra \tilde{A} is a $\widehat{\mathbb{P}}_n$ -algebra \tilde{M} together with a coassociative left coaction map $\tilde{M} \rightarrow \tilde{A} \otimes \tilde{M}$ which is a map of $\widehat{\mathbb{P}}_n$ -algebras.

Proposition 1.8. Thus defined differential, coaction, multiplication and L_∞ operations make $T_\bullet(A[1]) \otimes M$ into a left $\widehat{\mathbb{P}}_n$ -comodule over $T_\bullet(A[1])$.

Proof. To prove compatibility of the L_∞ operations with the coaction, it is enough to assume each argument is either in M or in $T_\bullet(A[1])$. If all arguments are in $T_\bullet(A[1])$, the compatibility with the coaction was checked in Proposition 1.6. If all arguments are in M and $k = 2$ we have

$$\Delta l_2([m_1], [m_2]) = [] \otimes [\{m_1, m_2\}]$$

and

$$l_2(\Delta([m_1]), \Delta([m_2])) = l_2([] \otimes [m_1], [] \otimes [m_2]) = [] \otimes [\{m_1, m_2\}].$$

If all but one arguments are in M and k is arbitrary we have

$$\begin{aligned}
& l_k(\Delta[a_1|\dots|a_p|1], \square \otimes [m_1], \dots, \square \otimes [m_{k-1}]) \\
&= \sum_{i=0}^p l_k([a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_p|1], \square \otimes [m_1], \dots, \square \otimes [m_{k-1}]) \\
&= \sum_{i=0}^p (-1)^{\sum_{q=1}^i |a_q|(1-(k-1)n)} [a_1|\dots|a_i] \otimes l_k([a_{i+1}|\dots|a_p|1], [m_1], \dots, [m_{k-1}]) \\
&= \sum_{i=0}^p (-1)^{\sum_{q=1}^p |a_q|(1-(k-1)n)} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|f_{k-1}(a_p; m_1, \dots, m_{k-1})]
\end{aligned}$$

and

$$\begin{aligned}
& \Delta l_k([a_1|\dots|a_p|1], [m_1], \dots, [m_{k-1}]) \\
&= (-1)^{\sum_{q=1}^p |a_q|(1-(k-1)n)} \Delta[a_1|\dots|a_{p-1}|f_{k-1}(a_p; m_1, \dots, m_{k-1})] \\
&= (-1)^{\sum_{q=1}^p |a_q|(1-(k-1)n)} \sum_{i=0}^p [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|f_{k-1}(a_p; m_1, \dots, m_{k-1})].
\end{aligned}$$

Therefore, as before it is enough to check symmetry, the Leibniz rule and Jacobi identity only after projecting to M . The operation l_k has a component in M if either all but one arguments are in M and one argument is in A or $k = 2$ and both arguments are in M .

- (Symmetry). Symmetry is clear for $l_2(m_1, m_2)$. For $l_k(a, m_1, \dots, m_{k-1})$ symmetry in the m_i variables follows from the symmetry property (2) of f_{k-1} .
- (Leibniz rule). If $k = 2$ we need to check that

$$l_2([m_1], [m_2 m_3]) = l_2([m_1], [m_2])[m_3] + (-1)^{|m_2||m_3|} l_2([m_1], [m_3])[m_2].$$

This is just an expression for the Leibniz rule in M . For any k we also need to check that

$$\begin{aligned}
l_k([a|1], [m_1], \dots, [m_{k-1} m_k]) &= l_k([a|1], [m_1], \dots, [m_{k-1}])[m_k] \\
&\quad + (-1)^{|m_{k-1}||m_k|} l_k([a|1], [m_1], \dots, [m_k])[m_{k-1}].
\end{aligned}$$

This immediately follows from the derivation property (3) of f_{k-1}

- (Jacobi identity).

The Jacobi identity has a component in M in the following four cases:

- (1) All arguments are in M . In this case we get the Jacobi identity for the bracket in M .
- (2) One argument is in A , the rest are in M .

The Jacobi identity is

$$\begin{aligned}
0 = & (-1)^{nk} l_{k+1}([da|1], [m_1], \dots, [m_k]) \\
& + dl_{k+1}([a|1], [m_1], \dots, [m_k]) \\
& + \sum_i (-1)^{|a| + \sum_{j=1}^{i-1} |m_j| + nk + 1} l_{k+1}([a|1], [m_1], \dots, [dm_i], \dots, [m_k]) \\
& + \sum_{i < j} (-1)^{|m_i| + \sum_{p=1}^{i-1} |m_p| + |m_j| + \sum_{p=1, p \neq i}^{j-1} |m_p| + n(i+j) + (|a|+1)(1-n)} l_k([a|1], \{[m_i], [m_j]\}, \dots) \\
& + \sum_i (-1)^{|m_i| + \sum_{j=i+1}^k |m_j| + in} \{l_k([a|1], [m_1], \dots, \widehat{[m_i]}, \dots, [m_k]), [m_i]\}.
\end{aligned}$$

Substituting l_k in terms of f_{k-1} from equation (9) we obtain

$$\begin{aligned}
0 = & (-1)^{nk} (-1)^{|a|(1-nk)} f_k(da; m_1, \dots, m_k) \\
& + (-1)^{(|a|+1)(1-nk)} df_k(a; m_1, \dots, m_k) \\
& + \sum_i (-1)^{|a| + \sum_{j=1}^{i-1} |m_j| + nk + (|a|+1)(1-nk)} f_k(a; m_1, \dots, dm_i, \dots, m_k) \\
& + \sum_{i < j} (-1)^{|m_i| + \sum_{p=1}^{i-1} |m_p| + |m_j| + \sum_{p=1, p \neq i}^{j-1} |m_p| + n(i+j) + (|a|+1)nk} f_{k-1}(a; \{m_i, m_j\}, \dots) \\
& + \sum_i (-1)^{|m_i| + \sum_{j=i+1}^k |m_j| + in + (|a|+1)(1-n(k-1))} \{f_{k-1}(a; m_1, \dots, \hat{m}_i, \dots, m_k), m_i\}.
\end{aligned}$$

After clearing out the signs, the equation coincides with (4).

(3) Two arguments are in A , the rest are in M .

The Jacobi identity is

$$\begin{aligned}
0 = & (-1)^{|a_1| + n + 1} l_{k+1}([\{a_1, a_2\}|1], [m_1], \dots, [m_k]) \\
& + \sum_{i+j=k+1} (-1)^{n(j+1)(k-j-1)} \sum_{\sigma \in S_{j, k-j}} \text{sgn}(\sigma)^n (-1)^{\epsilon_m} (-1)^{nj + (|a_1|+1)(1+nj)} \times \\
& \quad l_{i+1}([a_1|1], l_{j+1}([a_2|1], [m_{\sigma(1)}], \dots, [m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]) \\
& + \sum_{i+j=k+1} (-1)^{n(j+1)(k-j-1)} \sum_{\sigma \in S_{j, k-j}} \text{sgn}(\sigma)^n (-1)^{\epsilon_m} (-1)^{n(j+1) + (|a_2|+1)(nj + |a_1|)} \times \\
& \quad l_{i+1}([a_2|1], l_{j+1}([a_1|1], [m_{\sigma(1)}], \dots, [m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]).
\end{aligned}$$

Substituting l_k in terms of f_{k-1} from equation (9) we obtain

$$\begin{aligned}
0 &= (-1)^{|a_1|+n+1} (-1)^{(|a_1|+|a_2|-n+1)(1-nk)} f_k(\{a_1, a_2\}; m_1, \dots, m_k) \\
&+ \sum_{i+j=k+1} (-1)^{n(j+1)(k-j-1)} \sum_{\sigma \in S_{j,k-j}} \text{sgn}(\sigma)^n (-1)^{\epsilon_m + (|a_1|+1)n(k+1) + (|a_2|+1)(1+nj) + nj} \times \\
&\quad f_i(a_1; f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)}), m_{\sigma(j+1)}, \dots, m_{\sigma(k)}) \\
&+ \sum_{i+j=k+1} (-1)^{n(j+1)(k-j-1)} \sum_{\sigma \in S_{j,k-j}} \text{sgn}(\sigma)^n (-1)^{\epsilon_m + (|a_2|+1)(1-n(k+1)+|a_1|) + (|a_1|+1)(1-nj) + n(j+1)} \times \\
&\quad f_i(a_2; f_j(a_1; m_{\sigma(1)}, \dots, m_{\sigma(j)}), m_{\sigma(j+1)}, \dots, m_{\sigma(k)}).
\end{aligned}$$

After rearranging the signs, we get (5).

(4) One argument is in $(A[1])^{\otimes 2}$, the rest are in M .

The Jacobi identity is

$$\begin{aligned}
0 &= (-1)^{nk} l_{k+1}(\text{d}[a_1|a_2|1], [m_1], \dots, [m_k]) + \text{d}l_{k+1}([a_1|a_2|1], [m_1], \dots, [m_k]) \\
&+ \sum_{\substack{i+j=k \\ i,j>0}} \sum_{\sigma \in S_{j,i}} \text{sgn}(\sigma)^n (-1)^{nk(j+1)+\epsilon} l_{i+1}(l_{j+1}([a_1|a_2|1], [m_{\sigma(1)}], \dots, [m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}])
\end{aligned}$$

The projection of each term to M is

$$\begin{aligned}
&l_{k+1}(\text{d}[a_1|a_2|1], [m_1], \dots, [m_k]) \\
&= (-1)^{|a_1|+1} l_{k+1}([a_1 a_2 | 1], [m_1], \dots, [m_k]) \\
&+ (-1)^{|a_1|+|a_2|} l_{k+1}([a_1 | f_0(a_2)], [m_1], \dots, [m_k]) \\
&= (-1)^{|a_1|+1+(|a_1|+|a_2|+1)(1-nk)} f_k(a_1 a_2; m_1, \dots, m_k) \\
&+ (-1)^{|a_1|+|a_2|(1+\sum_{i=1}^k |m_i|) + (|a_1|+1)(1-nk)} f_k(a_1; m_1, \dots, m_k) f_0(a_2), \\
&\text{d}l_{k+1}([a_1|a_2|1], [m_1], \dots, [m_k]) = (-1)^{|a_1|+1+(|a_1|+|a_2|)(1-nk)} f_0(a_1) f_k(a_2; m_1, \dots, m_k), \\
&l_{i+1}(l_{j+1}([a_1|a_2|1], [m_{\sigma(1)}], \dots, [m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]) \\
&= (-1)^{(|a_1|+|a_2|)(1-nj)} l_{i+1}([a_1 | f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)}]), [m_{\sigma(j+1)}], \dots, [m_{\sigma(k)}]) \\
&= (-1)^{(|a_1|+|a_2|)(1-nj) + (|a_2|+\sum_{i=1}^j |m_{\sigma(i)}| + nj) \sum_{p=1}^{k-j} |m_{\sigma(j+p)}| + (|a_1|+1)(1-ni)} \times \\
&\quad f_i(a_1; m_{\sigma(j+1)}, \dots, m_{\sigma(k)}) f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)})
\end{aligned}$$

Let us denote by $\bar{\sigma} \in S_{i,j}$ the shuffle obtained from σ by swapping the blocks $\sigma(1), \dots, \sigma(j)$ and $\sigma(j+1), \dots, \sigma(k)$. That is, $\bar{\sigma}(p) = \sigma(j+p)$ for $1 \leq p \leq i$ and $\bar{\sigma}(p) = \sigma(p-i)$ for $i < p \leq k$. Denote by $\bar{\epsilon}$ the Koszul sign corresponding to the shuffle $\bar{\sigma}$. We have

$$\text{sgn}(\bar{\sigma}) = \text{sgn}(\sigma) (-1)^{j(k-j)}$$

and

$$(-1)^{\bar{\epsilon}} = (-1)^{\epsilon} (-1)^{\sum_{i=1}^j |m_{\sigma(i)}| \sum_{p=1}^{k-j} |m_{\sigma(j+p)}|}.$$

The Jacobi identity becomes

$$\begin{aligned}
0 &= (-1)^{|a_1|+(|a_1|+|a_2|)(1-nk)} f_k(a_1 a_2; m_1, \dots, m_k) \\
&\quad - (-1)^{|a_1|+|a_2|(1+\sum_{i=1}^k |m_i|)+|a_1|(1-nk)} f_k(a_1; m_1, \dots, m_k) f_0(a_2) \\
&\quad - (-1)^{|a_1|+(|a_1|+|a_2|)(1-nk)} f_0(a_1) f_k(a_2; m_1, \dots, m_k) \\
&\quad + \sum_{i+j=k; i,j>0} \sum_{\bar{\sigma} \in S_{i,j}} \text{sgn}(\bar{\sigma})^n (-1)^{(|a_1|+|a_2|)(1-nj)+(|a_2|+n j) \sum_{p=1}^{k-j} |m_{\sigma(j+p)}|+(|a_1|+1)(1-ni)+n(j+k)} \times \\
&\quad f_i(a_1; m_{\sigma(j+1)}, \dots, m_{\sigma(k)}) f_j(a_2; m_{\sigma(1)}, \dots, m_{\sigma(j)}).
\end{aligned}$$

Rearranging the signs, we obtain (6). □

In the same way we can make $M \otimes T_\bullet(A[1])$ into a $\widehat{\mathbb{P}}_n$ -algebra compatibly with the right coaction of $T_\bullet(A[1])$. The bar differential on $M \otimes T_\bullet(A[1])$ is given by

$$\begin{aligned}
d[m|a_1|\dots|a_n] &= [dm|a_1|\dots|a_n] \\
&\quad + \sum_{i=1}^n (-1)^{\sum_{q=1}^{i-1} |a_q|+i-1+|m|} [m|a_1|\dots|da_i|\dots|a_n] \\
&\quad + (-1)^{|m|+|a_1|+1} [ma_1|\dots|a_n] \\
&\quad + \sum_{i=1}^{n-1} (-1)^{\sum_{q=1}^i |a_q|+i+|m|} [m|a_1|\dots|a_i a_{i+1}|\dots|a_n].
\end{aligned}$$

Moreover, $M \otimes T_\bullet(A[1])$ is isomorphic to $T_\bullet(A^{\text{op}}[1])^{\text{cop}} \otimes M^{\text{op}}$ as right $T_\bullet(A[1])$ -comodules using the isomorphism (8). Here M^{op} represents the same cdga with the opposite bracket and the coisotropic structure given by $f_k^{\text{op}} = (-1)^k f_k$. Using the previous theorem, we can make $M \otimes T_\bullet(A[1])$ into a right $\widehat{\mathbb{P}}_n$ -comodule over $T_\bullet(A[1])$.

Let us now combine left and right comodules.

Theorem 1.9. *Let A be a \mathbb{P}_{n+1} -algebra and $A \rightarrow M$ and $A \rightarrow N$ two coisotropic morphisms. Then the two-sided bar complex $N \otimes_A^{\mathbb{L}} M$ has a natural structure of a $\widehat{\mathbb{P}}_n$ -algebra such that the natural projection $N^{\text{op}} \otimes M \rightarrow N \otimes_A^{\mathbb{L}} M$ is a Poisson morphism.*

Proof. Let $\tilde{A} = T_\bullet(A[1])$, $\tilde{N} = N \otimes \tilde{A}$ and $\tilde{M} = \tilde{A} \otimes M$. Then \tilde{A} is a \mathbb{P}_n -bialgebra, \tilde{N} a right $\widehat{\mathbb{P}}_n$ -comodule and \tilde{M} a left $\widehat{\mathbb{P}}_n$ -comodule over \tilde{A} .

We will first show that the cotensor product $\tilde{N} \otimes^{\tilde{A}} \tilde{M}$ is closed under the $\widehat{\mathbb{P}}_n$ -structures coming from $\tilde{N} \otimes \tilde{M}$.

Recall that

$$\tilde{N} \otimes^{\tilde{A}} \tilde{M} := \text{eq}(\tilde{N} \otimes \tilde{M} \rightrightarrows \tilde{N} \otimes \tilde{A} \otimes \tilde{M}),$$

where the two maps are coactions on \tilde{M} and \tilde{N} .

By definition the coaction

$$\tilde{M} \xrightarrow{\Delta_M} \tilde{A} \otimes \tilde{M}$$

is a map of $\widehat{\mathbb{P}}_n$ -algebras, so

$$\tilde{N} \otimes \tilde{M} \xrightarrow{\text{id}_{\tilde{N}} \otimes \Delta_M} \tilde{N} \otimes \tilde{A} \otimes \tilde{M}$$

is also a map of $\widehat{\mathbb{P}}_n$ -algebras. But it is obvious that an equalizer of maps of $\widehat{\mathbb{P}}_n$ -algebras is a $\widehat{\mathbb{P}}_n$ -algebra.

To conclude the proof of the theorem, observe that

$$\tilde{N} \otimes^{\tilde{A}} \tilde{M} \cong N \otimes_A^{\mathbb{L}} M.$$

Indeed,

$$N \otimes_A^{\mathbb{L}} M = N \otimes T_{\bullet}(A[1]) \otimes M \xrightarrow{\text{id}_N \otimes \Delta \otimes \text{id}_M} N \otimes T_{\bullet}(A[1]) \otimes T_{\bullet}(A[1]) \otimes M$$

lands in $\tilde{N} \otimes^{\tilde{A}} \tilde{M}$ due to coassociativity of $T_{\bullet}(A[1])$ and it is obviously an isomorphism. \square

2. CLASSICAL HAMILTONIAN REDUCTION

Let \mathfrak{g} be a finite-dimensional dg Lie algebra over k . In this section we apply results of the previous section to the \mathbb{P}_2 -algebra $A = C^{\bullet}(\mathfrak{g}, \text{Sym } \mathfrak{g})$. The results of this section generalize in a straightforward way to n -shifted Hamiltonian reduction in which case we replace A by the \mathbb{P}_{n+2} -algebra $C^{\bullet}(\mathfrak{g}, \text{Sym}(\mathfrak{g}[-n]))$.

2.1. Chevalley-Eilenberg complex. Let V be a \mathfrak{g} -representation. The Chevalley–Eilenberg complex $C^{\bullet}(\mathfrak{g}, V)$ is defined to be

$$C^{\bullet}(\mathfrak{g}, V) = \text{Hom}(\text{Sym}(\mathfrak{g}[1]), V)$$

with the differential

$$\begin{aligned} (df)(x_1, \dots, x_n) &= df(x_1, \dots, x_n) \\ &+ \sum_{i=1}^n (-1)^{\sum_{p=1}^{i-1} |x_p| + |f| + n + 1} f(x_1, \dots, dx_i, \dots, x_n) \\ &+ \sum_{i < j} (-1)^{|x_i| \sum_{p=1}^{i-1} |x_p| + |x_j| \sum_{p=1, p \neq i}^{j-1} |x_p| + i + j + |f|} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n) \\ (11) \quad &+ \sum_i (-1)^{|x_i|(\sum_{p=1}^{i-1} |x_p| + |f| + n + 1) + |f| + i + 1} x_i f(x_1, \dots, \widehat{x}_i, \dots, x_n). \end{aligned}$$

Here $|f|$ is the degree of f in $\text{Hom}(\text{Sym}(\mathfrak{g}[1]), V)$ and we have used the décalage isomorphism as in (1) to identify $\text{Hom}(\text{Sym}(\mathfrak{g}[1]), -)$ with antisymmetric functions on \mathfrak{g} .

The product

$$(12) \quad C^{\bullet}(\mathfrak{g}, A) \otimes C^{\bullet}(\mathfrak{g}, B) \rightarrow C^{\bullet}(\mathfrak{g}, A \otimes B)$$

is defined to be

$$(v \smile w)(x_1, \dots, x_{k+l}) = \sum_{\sigma \in S_{k,l}} \text{sgn}(\sigma) (-1)^{\epsilon_1 + \epsilon_l} v(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \otimes w(x_{\sigma(k+1)}, \dots, x_{\sigma(k+l)}),$$

where the sign is

$$\epsilon_1 = |w|k + \sum_{i=1}^k |x_{\sigma(i)}|(l + |w|).$$

The algebra $\text{Sym } \mathfrak{g}$ has the Kirillov–Kostant Poisson structure given on the generators by $\pi(x_1, x_2) = [x_1, x_2]$ for $x_i \in \mathfrak{g}$. The center of this \mathbb{P}_1 -algebra can be computed to be

$$Z(\text{Sym } \mathfrak{g}) \cong C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g})$$

with the bracket

$$\begin{aligned} [v, w](x_1, \dots, x_{k+l-1}) &= \sum_{\sigma \in S_{l, k-1}} \text{sgn}(\sigma) (-1)^{\epsilon_1 + \epsilon_2} v(w(x_{\sigma(1)}, \dots, x_{\sigma(l)}), x_{\sigma(l+1)}, \dots, x_{\sigma(k+l-1)}) \\ &\quad - \sum_{\sigma \in S_{k, l-1}} \text{sgn}(\sigma) (-1)^{\epsilon_1 + \epsilon_2} w(v(x_{\sigma(1)}, \dots, x_{\sigma(k)}), x_{\sigma(k+1)}, \dots, x_{\sigma(k+l-1)}), \end{aligned}$$

where $(-1)^\epsilon$ denotes the sign coming from the Koszul sign rule applied to the permutation σ of x_i and the signs ϵ_i are

$$\begin{aligned} \epsilon_1 &= (|w| + l)(k + 1) + |v| \\ \epsilon_2 &= (|v| - k)(|w| - l) + (k + 1)(|w| + 1) + |v|. \end{aligned}$$

2.2. Hamiltonian reduction. Let B be a \mathbb{P}_1 -algebra with a \mathfrak{g} -action preserving the Poisson bracket. We denote by $a: \mathfrak{g} \rightarrow \text{Der}(B)$ the action map.

Definition 2.1. A \mathfrak{g} -equivariant morphism of complexes $\mu: \mathfrak{g} \rightarrow B$ is a *moment map* for the \mathfrak{g} -action on B if the equation

$$\{\mu(x), b\} = a(x).b$$

is satisfied for all $x \in \mathfrak{g}$ and $b \in B$. In this case we say that the \mathfrak{g} -action is *Hamiltonian*.

Remark. One can replace \mathfrak{g} -equivariance in the definition of the moment map with the condition that the induced map $\text{Sym } \mathfrak{g} \rightarrow B$ is a \mathbb{P}_1 -map.

We define the Hamiltonian reduction to be

$$B // \text{Sym } \mathfrak{g} := C^\bullet(\mathfrak{g}, k) \otimes_{C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g})}^{\mathbb{L}} C^\bullet(\mathfrak{g}, B).$$

We will introduce a $\widehat{\mathbb{P}}_1$ -structure on this complex later in Corollary 2.3. Let us just mention a different complex used in derived Hamiltonian reduction called the classical BRST complex [KS87]

$$C^\bullet(\mathfrak{g}, \text{Sym}(\mathfrak{g}[1]) \otimes B).$$

Here the differential on $\text{Sym}(\mathfrak{g}[1]) \otimes B$ is the Koszul differential: given

$$x_1 \wedge \dots \wedge x_n \otimes b \in \text{Sym}(\mathfrak{g}[1]) \otimes B$$

we let

$$\begin{aligned} d(x_1 \wedge \dots \wedge x_n \otimes b) &= \sum_{i=1}^n (-1)^{(|x_i|+1)(\sum_{q=1}^{i-1} |x_q|+i-1)} dx_i \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_n \otimes b \\ &\quad - \sum_{i=1}^n (-1)^{|x_i| \sum_{q=i+1}^n (|x_q|+1) + \sum_{q=1}^{i-1} (|x_q|+1) + |x_i|} x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_n \otimes \mu(x_i)b \\ &\quad + (-1)^{\sum_{q=1}^n |x_q| + n} x_1 \wedge \dots \wedge x_n \otimes db. \end{aligned}$$

One can introduce a Lie bracket on the classical BRST complex as follows. As a commutative graded algebra, the classical BRST complex is generated by $\mathfrak{g}^*[-1]$, $\mathfrak{g}[1]$ and B . We keep the bracket on B and let the bracket between an element $\phi \in \mathfrak{g}^*[-1]$ and an element $x \in \mathfrak{g}[1]$ be the natural pairing: $\{\phi, x\} := \phi(x)$. Then d is a derivation of the bracket precisely due to the moment map equation.

2.3. Hamiltonian reduction as a coisotropic intersection. As a plain graded commutative algebra, $C^\bullet(\mathfrak{g}, B) \cong B \otimes \text{Sym}(\mathfrak{g}^*[-1])$, so its module of derivations is isomorphic to

$$T_B \otimes \text{Sym}(\mathfrak{g}^*[-1]) \oplus B \otimes \text{Sym}(\mathfrak{g}^*[-1]) \otimes \mathfrak{g}[1]$$

with the extra differential $\mathfrak{g} \rightarrow T_B$ given by the action map. Therefore, the Poisson center of $C^\bullet(\mathfrak{g}, B)$ is

$$Z(C^\bullet(\mathfrak{g}, B)) \cong C^\bullet(\mathfrak{g}, \widehat{\text{Sym}}(T_B[-1]) \otimes \widehat{\text{Sym}}(\mathfrak{g})).$$

Given a Hamiltonian \mathfrak{g} -action on B , let us define the morphism

$$C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g}) \rightarrow Z(C^\bullet(\mathfrak{g}, B))$$

as follows. The cdga $C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g})$ is generated by $C^\bullet(\mathfrak{g}, k)$ and $\mathfrak{g} \subset \text{Sym } \mathfrak{g}$. We let

$$C^\bullet(\mathfrak{g}, k) \hookrightarrow C^\bullet(\mathfrak{g}, \widehat{\text{Sym}}(T_B[-1]) \otimes \widehat{\text{Sym}}(\mathfrak{g}))$$

be the natural embedding. The map

$$\mathfrak{g} \rightarrow C^\bullet(\mathfrak{g}, \widehat{\text{Sym}}(T_B[-1]) \otimes \widehat{\text{Sym}}(\mathfrak{g}))$$

is given by $x \mapsto \mu(x) - x$ for $v \in \mathfrak{g}$.

Proposition 2.2. *Let B be a \mathbb{P}_1 -algebra with a Hamiltonian \mathfrak{g} -action. Then the morphism*

$$\mu: C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g}) \rightarrow C^\bullet(\mathfrak{g}, B)$$

is coisotropic.

Proof. It is enough to check that the morphism we have defined on generators commutes with the differential and the brackets.

Indeed, it is clear that the embedding $C^\bullet(\mathfrak{g}, k) \hookrightarrow Z(C^\bullet(\mathfrak{g}, B))$ commutes with differentials. For $x \in \mathfrak{g}$

$$d\mu(x) + [\pi, \mu(x)] - dx - (-1)^{|x|}a(x) = d\mu(x) - dx = \mu(dx) - dx,$$

where in the first equality we have used the moment map equation

$$[\pi, \mu(x)](b) = (-1)^{|x|}\{\mu(x), b\} = (-1)^{|x|}a(x).b.$$

It is also clear that the morphism commutes with brackets as B Poisson-commutes with $C^\bullet(\mathfrak{g}, \text{Sym}(\mathfrak{g})) \hookrightarrow Z(C^\bullet(\mathfrak{g}, B))$. \square

Example. Let $B = k$ with the trivial \mathfrak{g} -action and $\mu = 0$.

The morphism $C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g}) \rightarrow C^\bullet(\mathfrak{g}, k)$ given by the counit $\text{Sym } \mathfrak{g} \rightarrow k$ possesses a coisotropic structure given by the composite of the antipode $S: \text{Sym } \mathfrak{g} \rightarrow \text{Sym } \mathfrak{g}$ with the completion map

$$C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g}) \xrightarrow{S} C^\bullet(\mathfrak{g}, \text{Sym } \mathfrak{g}) \rightarrow Z(C^\bullet(\mathfrak{g}, k)) \cong C^\bullet(\mathfrak{g}, \widehat{\text{Sym}}(\mathfrak{g})).$$

Corollary 2.3. *The Poisson reduction*

$$B//\mathrm{Sym}\mathfrak{g} = C^\bullet(\mathfrak{g}, k) \otimes_{C^\bullet(\mathfrak{g}, \mathrm{Sym}\mathfrak{g})}^{\mathbb{L}} C^\bullet(\mathfrak{g}, B)$$

carries a natural $\widehat{\mathbb{P}}_1$ -structure. Moreover, it is quasi-isomorphic to the classical BRST complex.

Proof. Combining Proposition 2.2 with Theorem 1.9, we see that

$$C^\bullet(\mathfrak{g}, k) \otimes_{C^\bullet(\mathfrak{g}, \mathrm{Sym}\mathfrak{g})}^{\mathbb{L}} C^\bullet(\mathfrak{g}, B)$$

carries a $\widehat{\mathbb{P}}_1$ -structure.

The two-sided bar complex $k \otimes_{\mathrm{Sym}\mathfrak{g}}^{\mathbb{L}} B$ is the geometric realization of the simplicial complex V_\bullet where

$$V_n = k \otimes (\mathrm{Sym}\mathfrak{g})^{\otimes n} \otimes B.$$

We also denote by W_\bullet^1 the simplicial complex whose geometric realization is

$$\mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes_{\mathrm{Sym}(\mathfrak{g}^*[-1])}^{\mathbb{L}} \mathrm{Sym}(\mathfrak{g}^*[-1])$$

and by W_\bullet^2 the constant simplicial complex with $W_0^2 = \mathrm{Sym}(\mathfrak{g}^*[-1])$.

The two-sided bar complex $C^\bullet(\mathfrak{g}, k) \otimes_{C^\bullet(\mathfrak{g}, \mathrm{Sym}\mathfrak{g})}^{\mathbb{L}} C^\bullet(\mathfrak{g}, B)$ is computed as the geometric realization of the simplicial complex $V_\bullet \otimes W_\bullet^1$ with the Chevalley-Eilenberg differential (11). The multiplication map gives a weak equivalence of simplicial complexes $W_\bullet^1 \rightarrow W_\bullet^2$ which extends to a weak equivalence of simplicial complexes $V_\bullet \otimes W_\bullet^1 \rightarrow V_\bullet \otimes W_\bullet^2$ which acts as the identity on V_\bullet . This implies that the multiplication map gives a weak equivalence

$$C^\bullet(\mathfrak{g}, k) \otimes_{C^\bullet(\mathfrak{g}, \mathrm{Sym}\mathfrak{g})}^{\mathbb{L}} C^\bullet(\mathfrak{g}, B) \rightarrow C^\bullet(\mathfrak{g}, k \otimes_{\mathrm{Sym}\mathfrak{g}}^{\mathbb{L}} B).$$

We have a weak equivalence of \mathfrak{g} -representations

$$\mathrm{Sym}(\mathfrak{g}[1]) \otimes B \rightarrow k \otimes_{\mathrm{Sym}\mathfrak{g}}^{\mathbb{L}} B$$

given by the symmetrization

$$x_1 \wedge \dots \wedge x_n \otimes b \mapsto \sum_{\sigma \in S_n} (-1)^\epsilon [x_{\sigma(1)} | \dots | x_{\sigma(n)} | b].$$

This gives a quasi-isomorphism

$$C^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B) \rightarrow C^\bullet(\mathfrak{g}, k \otimes_{\mathrm{Sym}\mathfrak{g}}^{\mathbb{L}} B).$$

Combining these two quasi-isomorphisms we obtain a quasi-isomorphism

$$B//\mathrm{Sym}\mathfrak{g} \rightarrow C^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[1]) \otimes B)$$

to the classical BRST complex. □

Remark. We have a splitting of the multiplication map

$$\mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes \mathrm{Sym}(\mathfrak{g}^*[-1])^{\otimes n} \otimes \mathrm{Sym}(\mathfrak{g}^*[-1]) \rightarrow \mathrm{Sym}(\mathfrak{g}^*[-1])$$

given by sending $x \mapsto x \otimes 1^{\otimes n} \otimes 1$.

For \mathfrak{g} an abelian Lie algebra this gives a splitting

$$C^\bullet(\mathfrak{g}, k \otimes_{\mathrm{Sym}\mathfrak{g}}^{\mathbb{L}} B) \rightarrow C^\bullet(\mathfrak{g}, k) \otimes_{C^\bullet(\mathfrak{g}, \mathrm{Sym}\mathfrak{g})}^{\mathbb{L}} C^\bullet(\mathfrak{g}, B) = B//\mathrm{Sym}\mathfrak{g}.$$

It is easy to check that the composite map

$$C^\bullet(\mathfrak{g}, \text{Sym}(\mathfrak{g}[1]) \otimes B) \rightarrow C^\bullet(\mathfrak{g}, k \otimes_{\text{Sym} \mathfrak{g}}^{\mathbb{L}} B) \rightarrow B // \text{Sym} \mathfrak{g}$$

is compatible with the Poisson structures.

3. BRACE ALGEBRAS

In this section we introduce quantum versions of \mathbb{P}_2 -algebras called brace algebras introduced by Gerstenhaber and Voronov, see [GV95] and [GV94]. By a theorem of McClure and Smith [MS99] the brace operad controlling brace algebras is a model of the \mathbb{E}_2 operad, i.e. the chain operad of little disks.

3.1. Algebras.

Definition 3.1. A *brace algebra* A is a dga together with brace operations $A \otimes A^{\otimes n} \rightarrow A[-n]$ for $n > 0$ denoted by $x\{y_1, \dots, y_n\}$ satisfying the following equations:

- (Associativity).

$$x\{y_1, \dots, y_n\}\{z_1, \dots, z_m\} = \sum (-1)^\epsilon x\{z_1, \dots, z_{i_1}, y_1\{z_{i_1+1}, \dots\}, \dots, y_n\{z_{i_n+1}, \dots\}, \dots, z_m\},$$

where the sum goes over the locations of the y_i insertions and the length of each y_i brace. The sign is

$$\epsilon = \sum_{p=1}^n (|y_p| + 1) \sum_{q=1}^{i_p} (|z_q| + 1).$$

- (Higher homotopies).

$$\begin{aligned} d(x\{y_1, \dots, y_n\}) &= (dx)\{y_1, \dots, y_n\} \\ &+ \sum_i (-1)^{|x| + \sum_{q=1}^{i-1} |y_q| + i} x\{y_1, \dots, dy_i, \dots, y_n\} \\ &+ \sum_i (-1)^{|x| + \sum_{q=1}^i |y_q| + i + 1} x\{y_1, \dots, y_i y_{i+1}, \dots, y_n\} \\ &- (-1)^{(|y_1| + 1)|x|} y_1 \cdot x\{y_2, \dots, y_n\} \\ &- (-1)^{|x| + \sum_{q=1}^{n-1} |y_q| + n} x\{y_1, \dots, y_{n-1}\} \cdot y_n. \end{aligned}$$

- (Distributivity).

$$\sum_{k=0}^n (-1)^{|x_2|(\sum_{q=1}^k |y_q| + k)} x_1\{y_1, \dots, y_k\} x_2\{y_{k+1}, \dots, y_n\} = (x_1 \cdot x_2)\{y_1, \dots, y_n\}.$$

In the axioms we use a shorthand notation $x\{\} \equiv x$.

Remark. These axioms coincide with the ones in [GV95] if one flips the sign of the differential.

For instance, the second axiom for $n = 1$ is equivalent to

$$xy - (-1)^{|x||y|} yx = (-1)^{|x|} d(x\{y\}) - (-1)^{|x|} (dx)\{y\} + x\{dy\}.$$

In other words, the multiplication is commutative up to homotopy.

One has the *opposite* brace algebra A^{op} defined as follows. The product on A^{op} is the opposite of that of A :

$$a \cdot^{\text{op}} b := (-1)^{|a||b|} b \cdot a$$

while the braces on A^{op} are defined by

$$x\{y_1, \dots, y_n\}^{\text{op}} = (-1)^{\sum_{i < j} (|y_i|+1)(|y_j|+1)+n} x\{y_n, \dots, y_1\}.$$

3.2. Modules. Let A be a brace algebra. We are now going to define modules over such algebras.

Definition 3.2. A *left brace A -module* is a dga M together with a left A -module structure and brace operations $M \otimes A^{\otimes n} \rightarrow M[-n]$ denoted by $m\{x_1, \dots, x_n\}$ satisfying the following equations:

- (Compatibility). For any $x, y_i \in A$ one has

$$(x \cdot 1)\{y_1, \dots, y_n\} = x\{y_1, \dots, y_n\} \cdot 1.$$

- (Associativity). For any $m \in M$ and $x_i, y_i \in A$ one has

$$m\{x_1, \dots, x_n\}\{y_1, \dots, y_m\} = \sum (-1)^\epsilon m\{y_1, \dots, y_{i_1}, x_1\{y_{i_1+1}, \dots\}, \dots, x_n\{y_{i_n+1}, \dots\}, \dots, y_m\},$$

where the sign is

$$\epsilon = \sum_{p=1}^n (|x_p| + 1) \sum_{q=1}^{i_p} (|y_q| + 1).$$

- (Higher homotopies). For any $m \in M$ and $x_i \in A$ one has

$$\begin{aligned} d(m\{x_1, \dots, x_n\}) &= (dm)\{x_1, \dots, x_n\} \\ &+ \sum (-1)^{|m| + \sum_{q=1}^{i-1} |x_q| + i} m\{x_1, \dots, dx_i, \dots, x_n\} \\ &+ \sum (-1)^{|m| + \sum_{q=1}^i |x_q| + i+1} m\{x_1, \dots, x_i x_{i+1}, \dots, x_n\} \\ &- (-1)^{|m|(|x_1|+1)} x_1 \cdot m\{x_2, \dots, x_n\} \\ &- (-1)^{|m| + \sum_{q=1}^{n-1} |x_q| + n} m\{x_1, \dots, x_{n-1}\} \cdot x_n. \end{aligned}$$

- (Distributivity). For any $m, n \in M$ and $x_i \in A$ one has

$$(mn)\{x_1, \dots, x_n\} = \sum_{k=0}^n (-1)^{|n|(\sum_{q=1}^k |x_q| + k)} m\{x_1, \dots, x_k\} n\{x_{k+1}, \dots, x_n\}.$$

Example. If A is a brace algebra, then it is a left A -module using the brace operations on A itself.

We define *right brace A -modules* to be left brace A^{op} -modules. If M is a left brace A -module, then M^{op} is naturally a right brace A -module with the brace operations mirror reversed.

3.3. Koszul duality. Let A be a brace algebra. Recall from Section 1.4 the bar complex $T_\bullet(A[1])$ which is a dg coalgebra. Since A is not commutative, the shuffle product is not compatible with the differential, so we introduce a slightly different product.

A product

$$T_\bullet(A[1]) \otimes T_\bullet(A[1]) \rightarrow T_\bullet(A[1])$$

is uniquely specified by the maps

$$A^{\otimes n} \otimes A^{\otimes m} \rightarrow A[1 - n - m].$$

We let the maps with $n = 1$ be given by the brace operations and the maps with $n \neq 1$ be zero. Our sign conventions are such that

$$[x] \cdot [y_1 | \dots | y_n] = [x\{y_1, \dots, y_n\}] + \dots$$

More generally, we have

$$[x_1 | \dots | x_n] \cdot [y_1 | \dots | y_m] = \sum_{\{i_p, l_p\}_{p=1}^n} (-1)^\epsilon [y_1 | \dots | y_{i_1} | x_1\{y_{i_1+1}, \dots, y_{i_1+l_1}\} | \dots | x_n\{y_{i_n+1}, \dots, y_{i_n+l_n}\} | \dots | y_m],$$

where the sign is

$$\epsilon = \sum_{p=1}^n (|x_p| + 1) \sum_{q=1}^{i_p} (|y_q| + 1).$$

Example. Let A be a commutative algebra considered as a brace algebra with vanishing brace operations. Then the product we have defined coincides with the shuffle product.

Proposition 3.3 (Gerstenhaber–Voronov). *Let A be a brace algebra. Thus defined multiplication on $T_\bullet(A[1])$ makes it into a dg bialgebra.*

Proof. By definition the product is compatible with the comultiplication and we only have to check associativity and the Leibniz rule for d .

It is enough to check the components of the identities landing in $A[1]$.

- (Associativity). The equation

$$([x] \cdot [y_1 | \dots | y_n]) \cdot [z_1 | \dots | z_m] = [x] \cdot ([y_1 | \dots | y_n] \cdot [z_1 | \dots | z_m])$$

has the following A component:

$$\begin{aligned} & x\{y_1, \dots, y_n\}\{z_1, \dots, z_m\} \\ &= \sum_{\{i_p, l_p\}_{p=1}^n} (-1)^\epsilon x\{z_1, \dots, z_{i_1}, y_1\{z_{i_1+1}, \dots, z_{i_1+l_1}\}, \dots, y_n\{z_{i_n+1}, \dots, z_{i_n+l_n}\}, \dots, z_m\}. \end{aligned}$$

This exactly coincides with the associativity property for brace algebras.

If we replace $[x]$ by $[x_1 | \dots | x_m]$ for $m > 1$, the associativity equation will have a trivial A component.

- (Derivation). The equation

$$d([x] \cdot [y_1 | \dots | y_n]) = [dx] \cdot [y_1 | \dots | y_n] + (-1)^{|x|+1} [x] \cdot d[y_1 | \dots | y_n]$$

has the following A component:

$$\begin{aligned} d(x\{y_1, \dots, y_n\}) + (-1)^{|x| + \sum_{q=1}^{n-1} |y_q| + n} x\{y_1, \dots, y_{n-1}\} \cdot y_n = \\ - (-1)^{(|y_1|+1)|x|} y_1 \cdot x\{y_2, \dots, y_n\} + (dx)\{y_1, \dots, y_n\} \\ + \sum_{i=1}^n (-1)^{\sum_{q=1}^{i-1} |y_q| + |x| + i} x\{y_1, \dots, dy_i, \dots, y_n\} \\ + \sum_{i=1}^{n-1} (-1)^{\sum_{q=1}^i |y_q| + |x| + i + 1} x\{y_1, \dots, y_i y_{i+1}, \dots, y_n\}. \end{aligned}$$

This follows from the higher homotopy identities for brace algebras.

The equation

$$d([x_1|x_2] \cdot [y_1|\dots|y_n]) = d[x_1|x_2] \cdot [y_1|\dots|y_n] + (-1)^{|x_1|+|x_2|} [x_1|x_2] \cdot d[y_1|\dots|y_n]$$

has the following A component:

$$\begin{aligned} \sum_{m=0}^n (-1)^{(|x_2|+1)(\sum_{q=1}^m |y_q|+m)} (-1)^{|x_1| + \sum_{q=1}^m |y_q| + m + 1} x_1\{y_1, \dots, y_m\} x_2\{y_{m+1}, \dots, y_n\} \\ = (-1)^{|x_1|+1} (x_1 x_2) \{y_1, \dots, y_n\}. \end{aligned}$$

This follows from the distributivity property for brace algebras.

If we instead have $[x_1|\dots|x_m]$ for $m > 2$, this equation will have a trivial A component.

□

Remark. It is not difficult to see that $T_\bullet(A[1])^{\text{cop}} \cong T_\bullet(A^{\text{op}}[1])$ under the isomorphism (8). Here $(\dots)^{\text{cop}}$ refers to the same dg algebra with the opposite coproduct and A^{op} is the opposite brace algebra.

Let us move on to a relative version of this statement. Let A be a brace algebra as before and M a left brace A -module. Recall the differential on the bar complex $T_\bullet(A[1]) \otimes M$. We are going to define a dg algebra structure on $T_\bullet(A[1]) \otimes M$ compatibly with the left coaction of $T_\bullet(A[1])$ such that M and $T_\bullet(A[1])$ are subalgebras. Thus, we just need to define a braiding morphism

$$M \otimes T_\bullet(A[1]) \rightarrow T_\bullet(A[1]) \otimes M.$$

Compatibility with the $T_\bullet(A[1])$ -comodule structure allows one to uniquely reconstruct this map from the composite

$$M \otimes T_\bullet(A[1]) \rightarrow T_\bullet(A[1]) \otimes M \rightarrow M.$$

We define it using the brace A -module structure on M . That is, the product is given by

$$[m] \cdot [x_1|\dots|x_n|1] = \sum_{i=0}^n (-1)^{|m|(\sum_{q=1}^i |x_q|+i)} [x_1|\dots|x_i|m\{x_{i+1}, \dots, x_n\}].$$

Proposition 3.4. *Let M be a left brace A -module. The previous formula defines a dga structure on $T_\bullet(A[1]) \otimes M$ compatibly with the left $T_\bullet(A[1])$ -comodule structure.*

We have the same statement for right brace A -modules. Indeed, one can replace A by A^{op} in the previous proposition and observe that the bar complexes $T_{\bullet}(A[1])^{\text{cop}} \otimes M$ and $M \otimes T_{\bullet}(A[1])$ are isomorphic.

We can combine left and right modules as follows.

Theorem 3.5. *Let A be a brace algebra, M a left brace A -module and N a right brace A -module. Then the intersection $N \otimes_A^{\mathbb{L}} M$ carries a natural dga structure so that the projection $N^{\text{op}} \otimes M \rightarrow N \otimes_A^{\mathbb{L}} M$ is a morphism of dg algebras.*

Proof. By Proposition 3.3 the bar complex $T_{\bullet}(A[1])$ is a dg bialgebra.

Now let $\tilde{M} = T_{\bullet}(A[1]) \otimes M$ and $\tilde{N} = N \otimes T_{\bullet}(A[1])$. By the previous proposition \tilde{M} is a left $T_{\bullet}(A[1])$ -comodule while \tilde{N} is a right $T_{\bullet}(A[1])$ -comodule.

The two-sided bar complex $N \otimes_A^{\mathbb{L}} M$ is isomorphic to the cotensor product $\tilde{N} \otimes^{T_{\bullet}(A[1])} \tilde{M}$. As both \tilde{N} and \tilde{M} are dg algebras which are compatible with the coaction of $T_{\bullet}(A[1])$, their cotensor product is also a dga. \square

4. QUANTUM HAMILTONIAN REDUCTION

4.1. Hochschild cohomology. Let A be a dga and B an A -bimodule. We define the Hochschild cochain complex $\text{CC}^{\bullet}(A, B)$ to be the graded vector space

$$\text{CC}^{\bullet}(A, B) = \bigoplus_{n=0}^{\infty} \text{Hom}(A^{\otimes n}, B)[-n]$$

with the differential

$$\begin{aligned} (df)(x_1, \dots, x_n) &= df(x_1, \dots, x_n) \\ &+ \sum_{i=1}^n (-1)^{|f| + \sum_{q=1}^{i-1} |x_q| + i + 1} f(x_1, \dots, dx_i, \dots, x_n) \\ &+ \sum_{i=1}^{n-1} (-1)^{|f| + \sum_{q=1}^i |x_q| + i} f(x_1, \dots, x_i x_{i+1}, \dots, x_n) \\ &+ (-1)^{|f|(|x_1|+1)} x_1 f(x_2, \dots, x_n) + (-1)^{\sum_{q=1}^{n-1} |x_q| + |f| + n} f(x_1, \dots, x_{n-1}) x_n. \end{aligned}$$

Given two A -bimodules B_1 and B_2 we have a cup product map

$$\text{CC}^{\bullet}(A, B_1) \otimes \text{CC}^{\bullet}(A, B_2) \rightarrow \text{CC}^{\bullet}(A, B_1 \otimes B_2)$$

given by

$$(f_1 \smile f_2)(x_1, \dots, x_n) = \sum_{i=0}^n (-1)^{|f_2|(\sum_{q=1}^i |x_q| + i)} f_1(x_1, \dots, x_i) \otimes f_2(x_{i+1}, \dots, x_n).$$

A relation between Hochschild and Chevalley–Eilenberg cohomology is given by the following construction. Let V be a $\mathfrak{U}\mathfrak{g}$ -bimodule. Then V^{ad} is a \mathfrak{g} -representation with the action given by

$$x.v := xv - (-1)^{|x||v|} vx, \quad x \in \mathfrak{g}, \quad v \in V.$$

Consider $f \in \text{Hom}((U\mathfrak{g})^{\otimes n}, V)[-n] \subset \text{CC}^\bullet(U\mathfrak{g}, V)$. We get an element $\tilde{f} \in \text{C}^\bullet(\mathfrak{g}, V)$ by the following formula:

$$\tilde{f}(x_1, \dots, x_n) = (-1)^{\sum_{q=1}^{n-1} (n-q)|x_q|} \sum_{\sigma \in S_n} (-1)^\epsilon f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

where ϵ is given by the Koszul sign rule with x_i in degree $|x_i| + 1$.

The following theorem can be found in [CR11, Theorem 2.5].

Proposition 4.1. *Let $A = U\mathfrak{g}$ and V be a $U\mathfrak{g}$ -bimodule. Then the morphism*

$$\text{CC}^\bullet(U\mathfrak{g}, V) \rightarrow \text{C}^\bullet(\mathfrak{g}, V)$$

we have defined is a quasi-isomorphism. Moreover, it is compatible with cup products.

4.2. Hochschild cohomology and braces. Gerstenhaber and Voronov [GV95] observed that the Hochschild cochain complex $\text{CC}^\bullet(A, A)$ is a brace algebra which was the motivating example. We define the brace operations as follows:

$$(13) \quad \begin{aligned} & x\{x_1, \dots, x_n\}(a_1, \dots, a_m) \\ &= \sum (-1)^\epsilon x(a_1, \dots, a_{i_1}, x_1(a_{i_1+1}, \dots, a_{i_1+l_1}), \dots, x_n(a_{i_n+1}, \dots, a_{i_n+l_n}), \dots, a_m), \end{aligned}$$

where the sign is determined by the following rule: x_i moving past a_j produces a sign $(|x_i| + 1)(|a_j| + 1)$.

A multiplication on A determines a degree 2 element m of $\text{CC}^\bullet(A, A)$ via

$$m(x, y) = (-1)^{|x|+1}xy.$$

The differential on $\text{CC}^\bullet(A, A)$ is the sum of the natural differential on $\oplus_n \text{Hom}((A[1])^{\otimes n}, A)$ and the differential $m\{f\} + (-1)^{|f|}f\{m\}$. The cup product on $\text{CC}^\bullet(A, A)$ is given by the formula $f_1 \smile f_2 = (-1)^{|f_1|+1}m\{f_1, f_2\}$.

We will also need a variation of this example. Let B be a dga and $\mu: A \rightarrow B$ a morphism. Using the brace operations as above, one can turn $\text{CC}^\bullet(A, B)$ into a left brace $\text{CC}^\bullet(A, A)$ -module.

We can also use the Hochschild cochain complex to give an interpretation of brace modules similar to Definition 1.4.

Proposition 4.2. *Let A be a brace algebra and M a left brace A -module with the module structure given by a morphism of algebras $f_0: A \rightarrow M$. Then we have a natural lift*

$$\begin{array}{ccc} T_\bullet(A[1]) & \xrightarrow{f_0} & T_\bullet(M[1]) \\ & \searrow f & \uparrow \\ & & T_\bullet(\text{CC}^\bullet(M, M)[1]), \end{array}$$

where f is a morphism of dg bialgebras.

Proof. A morphism of coalgebras $f: T_\bullet(A[1]) \rightarrow T_\bullet(\text{CC}^\bullet(M, M)[1])$ is uniquely specified by the composite $T_\bullet(A[1]) \rightarrow T_\bullet(\text{CC}^\bullet(M, M)[1]) \rightarrow \text{CC}^\bullet(M, M)[1]$ which consists of morphisms

$$f_{m,n}: M^{\otimes m} \otimes A^{\otimes n} \rightarrow M[1 - n - m].$$

We define $f_{m,n} = 0$ for $m > 1$. The operations $f_{1,n}$ are given by

$$f_{1,n}(m, x_1, \dots, x_n) = m\{x_1, \dots, x_n\}.$$

A straightforward computation shows that the first two axioms in Definition 3.2 are equivalent to the compatibility of f with the multiplications and the last two axioms are equivalent to the compatibility of f with the differentials. \square

Remark. A triple (A, M, f) of a brace algebra A , a dga M and a morphism of brace algebras $f: A \rightarrow \text{CC}^\bullet(M, M)$ is the same as an algebra over the two-dimensional Swiss-cheese operad, see [DTT09] and [Th10].

4.3. Hamiltonian reduction. Let B be a dg algebra with a \mathfrak{g} -action. We denote by $a: \mathfrak{g} \rightarrow \text{Der}(B)$ the action morphism.

Definition 4.3. A \mathfrak{g} -equivariant morphism $\mu: \mathfrak{g} \rightarrow B$ is a *quantum moment map* if the equation

$$[\mu(x), b] = a(x).b$$

is satisfied for all $x \in \mathfrak{g}$ and $b \in B$.

Remark. As in the case of classical moment maps, one can replace \mathfrak{g} -equivariance by the condition that μ extends to a morphism of dg algebras $U\mathfrak{g} \rightarrow B$.

We define quantum Hamiltonian reduction $B//U\mathfrak{g}$ by

$$B//U\mathfrak{g} = \text{CC}^\bullet(U\mathfrak{g}, k) \otimes_{\text{CC}^\bullet(U\mathfrak{g}, U\mathfrak{g})}^{\mathbb{L}} \text{CC}^\bullet(U\mathfrak{g}, B).$$

In this bar complex we use the left $\text{CC}^\bullet(U\mathfrak{g}, U\mathfrak{g})$ -module structure on $\text{CC}^\bullet(U\mathfrak{g}, B)$ coming from the moment map $U\mathfrak{g} \rightarrow B$ and the right $\text{CC}^\bullet(U\mathfrak{g}, U\mathfrak{g})$ -module structure on $\text{CC}^\bullet(U\mathfrak{g}, k)$ coming from the counit. We put a dga structure on $B//U\mathfrak{g}$ in Corollary 4.4.

There is a quantum version of the BRST complex. As a complex, it has the following description. We will assume that the Lie algebra \mathfrak{g} is unimodular, i.e. the representation $\det(\mathfrak{g})$ is trivial.

Recall the Koszul complex $\text{Sym}(\mathfrak{g}[1]) \otimes B$ that we have defined in Section 2.2. We are going to deform it to the Chevalley–Eilenberg differential as follows. Given

$$x_1 \wedge \dots \wedge x_n \otimes b \in \text{Sym}(\mathfrak{g}[1]) \otimes B$$

we let

$$\begin{aligned} d(x_1 \wedge \dots \wedge x_n \otimes b) &= \sum_{i=1}^n (-1)^{(|x_i|+1)(\sum_{q=1}^{i-1} |x_q|+i-1)} dx_i \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_n \otimes b \\ &\quad - \sum_{i=1}^n (-1)^{|x_i| \sum_{q=i+1}^n (|x_q|+1) + \sum_{q=1}^{i-1} (|x_q|+1) + |x_i|} x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_n \otimes \mu(x_i)b \\ &\quad + (-1)^{\sum_{q=1}^n |x_q| + n} x_1 \wedge \dots \wedge x_n \otimes db \\ &\quad + \sum_{i < j} (-1)^{(|x_i|+1)(\sum_{q=1}^{i-1} |x_q|+i) + (|x_j|+1)(\sum_{q=1, q \neq i}^{j-1} |x_q|+j-1)} \times \\ (14) \quad &\quad \times [x_i, x_j] \wedge x_1 \wedge \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n \otimes b. \end{aligned}$$

The quantum BRST complex is then

$$C^\bullet(\mathfrak{g}, \text{Sym}(\mathfrak{g}[1]) \otimes B).$$

We refer the reader to [KS87, Section 6] for a detailed description of the quantum BRST complex together with a dga structure.

4.4. Hamiltonian reduction as an intersection. Let B be a dga with a Hamiltonian action of \mathfrak{g} . Recall that $CC^\bullet(U\mathfrak{g}, B)$ is then a left brace module over $CC^\bullet(U\mathfrak{g}, U\mathfrak{g})$. Similarly, $CC^\bullet(U\mathfrak{g}, k)$ is a left brace module using the counit map $U\mathfrak{g} \rightarrow k$ and hence $CC^\bullet(U\mathfrak{g}, k)^{\text{op}}$ is a right brace module. Using Theorem 3.5 we therefore have a natural multiplication on the tensor product of $CC^\bullet(U\mathfrak{g}, k)$ and $CC^\bullet(U\mathfrak{g}, B)$.

Corollary 4.4. *The quantum Hamiltonian reduction*

$$B//U\mathfrak{g} = CC^\bullet(U\mathfrak{g}, k) \otimes_{CC^\bullet(U\mathfrak{g}, U\mathfrak{g})}^{\mathbb{L}} CC^\bullet(U\mathfrak{g}, B)$$

carries a natural dga structure. Moreover, there is a zig-zag of quasi-isomorphisms with the quantum BRST complex.

Proof. The zig-zag of quasi-isomorphisms mentioned in the statement of the theorem is as follows:

$$\begin{array}{c} CC^\bullet(U\mathfrak{g}, k) \otimes_{CC^\bullet(U\mathfrak{g}, U\mathfrak{g})}^{\mathbb{L}} CC^\bullet(U\mathfrak{g}, B) \\ \downarrow \\ CC^\bullet(U\mathfrak{g}, k \otimes_{U\mathfrak{g}}^{\mathbb{L}} B) \\ \uparrow \\ CC^\bullet(U\mathfrak{g}, \text{Sym}(\mathfrak{g}[1]) \otimes B) \\ \downarrow \\ C^\bullet(\mathfrak{g}, \text{Sym}(\mathfrak{g}[1]) \otimes B). \end{array}$$

- The morphism

$$CC^\bullet(U\mathfrak{g}, k) \otimes_{CC^\bullet(U\mathfrak{g}, U\mathfrak{g})}^{\mathbb{L}} CC^\bullet(U\mathfrak{g}, B) \rightarrow CC^\bullet(U\mathfrak{g}, k \otimes_{U\mathfrak{g}}^{\mathbb{L}} B)$$

is given by the cup product. The fact that it is a quasi-isomorphism is proved as in Corollary 2.3.

- The morphism

$$CC^\bullet(U\mathfrak{g}, \text{Sym}(\mathfrak{g}[1]) \otimes B) \hookrightarrow CC^\bullet(U\mathfrak{g}, k \otimes_{U\mathfrak{g}}^{\mathbb{L}} B)$$

is given by including the Chevalley–Eilenberg chain complex into the bar complex.

- The morphism

$$CC^\bullet(U\mathfrak{g}, \text{Sym}(\mathfrak{g}[1]) \otimes B) \rightarrow C^\bullet(\mathfrak{g}, \text{Sym}(\mathfrak{g}[1]) \otimes B)$$

is the restriction morphism which is a quasi-isomorphism by Proposition 4.1.

□

4.5. \mathbb{E}_n Hamiltonian reduction. The interpretation of quantum Hamiltonian reduction as a tensor product of brace modules allows one to formulate an \mathbb{E}_n version of quantum Hamiltonian reduction. We will briefly remind the reader the basics of \mathbb{E}_n -algebras referring to [Gin13] for a comprehensive treatment.

Let \mathbb{E}_n be the chain operad of little n -cubes. For instance, the operad \mathbb{E}_1 is quasi-isomorphic to the associative operad and \mathbb{E}_2 is quasi-isomorphic to the brace operad. One can think of \mathbb{E}_n -algebras as locally-constant factorization algebras on \mathbb{R}^n valued in chain complexes.

Given a morphism of \mathbb{E}_n -algebras $f: A \rightarrow B$ one has the \mathbb{E}_n -centralizer $Z(f)$ which is an \mathbb{E}_n -algebra satisfying a certain universal property [Gin13, Definition 24]. For $f = \text{id}: A \rightarrow A$ we denote $Z(\text{id}) = Z(A)$, the center of A which is an $\mathbb{E}_1 \otimes_{BV} \mathbb{E}_n \cong \mathbb{E}_{n+1}$ -algebra.

Given an \mathbb{E}_n -algebra A , one has an induced Lie algebra structure of degree $1 - n$. The left adjoint to the forgetful functor is called the universal enveloping \mathbb{E}_n -algebra functor and is denoted by $U_{\mathbb{E}_n}$.

Let B be an \mathbb{E}_n -algebra with an action of the Lie algebra \mathfrak{g} , i.e. we have a morphism of Lie algebras $a: \mathfrak{g} \rightarrow \mathbb{T}_B$ to the tangent complex of B .

Definition 4.5. A morphism of \mathbb{E}_n -algebras $\mu: U_{\mathbb{E}_n} \mathfrak{g} \rightarrow B$ is called the *quantum moment map* for the \mathfrak{g} -action if the diagram

$$\begin{array}{ccc} B[n-1] & \longrightarrow & \mathbb{T}_B \\ \mu \uparrow & & \uparrow a \\ U_{\mathbb{E}_n}[n-1] & \longleftarrow & \mathfrak{g} \end{array}$$

is commutative.

Given a moment map μ we get that $Z(\mu)$ is a left module over the $\mathbb{E}_1 \otimes \mathbb{E}_n$ -algebra $Z(U_{\mathbb{E}_n} \mathfrak{g})$ in the ∞ -category of \mathbb{E}_n -algebras. Note that using [Gin13, Theorem 14] one can identify $Z(\mu) \cong C^\bullet(\mathfrak{g}, B)$.

Let $\epsilon: U_{\mathbb{E}_n} \mathfrak{g} \rightarrow k$ be the counit map. If $n > 0$ one can choose an isomorphism

$$Z(U_{\mathbb{E}_n} \mathfrak{g}) \cong Z(U_{\mathbb{E}_n} \mathfrak{g})^{\text{op}}$$

making $Z(\epsilon)$ into a right module over $Z(U_{\mathbb{E}_n} \mathfrak{g})$.

Therefore, the \mathbb{E}_n Hamiltonian reduction

$$B // U_{\mathbb{E}_n} \mathfrak{g} = Z(\epsilon) \otimes_{Z(U_{\mathbb{E}_n} \mathfrak{g})}^{\mathbb{L}} Z(\mu)$$

is an \mathbb{E}_n -algebra.

Unwinding the definitions for $n = 1$ and identifying the $\mathbb{E}_1 \otimes_{BV} \mathbb{E}_1$ -structure on the Hochschild complex $Z(A) \cong CC^\bullet(A, A)$ of a dga A with the explicit brace structure we see that \mathbb{E}_1 Hamiltonian reduction is the same as the usual quantum Hamiltonian reduction.

5. CLASSICAL LIMITS

In this section we relate some constructions in Section 1 to those in Section 3. Namely, we formulate precisely in which sense constructions in Section 3 are quantizations. Along the way we also relate Baranovsky and Ginzburg's construction [BG09] of the Poisson structure on a coisotropic intersection to our formulas.

5.1. Beilinson–Drinfeld algebras. A precise sense in which associative algebras are quantizations of Poisson algebras is given by Beilinson–Drinfeld (BD) algebras [CG15, Section 8.4]. Let us recall the definition.

Definition 5.1. A \mathbb{BD}_1 -algebra is a dgla A over $k[[\hbar]]$ together with an associative multiplication satisfying the relations

- $\hbar\{x, y\} = xy - (-1)^{|x||y|}yx,$
- $\{x, yz\} = \{x, y\}z + (-1)^{|x||y|}y\{x, z\}.$

To understand this definition, recall that dg algebras are naturally Lie algebras with the bracket given by the commutator. The notion of a \mathbb{BD}_1 -algebra then captures the fact that the Lie bracket vanishes to the first order at $\hbar = 0$. In the classical limit we have an isomorphism of operads

$$\mathbb{BD}_1/\hbar \cong \mathbb{P}_1$$

while in the quantum case $\hbar \neq 0$ we have

$$\mathbb{BD}_1[\hbar^{-1}] \cong \text{Ass} \otimes_k k((\hbar))$$

since the bracket is then uniquely determined from the multiplication. In other words, the operad \mathbb{BD}_1 interpolates between the Poisson operad \mathbb{P}_1 and the associative operad Ass .

Let us also mention that there is a canonical isomorphism of operads

$$\mathbb{P}_1 \otimes k[\hbar]/\hbar^2 \xrightarrow{\sim} \mathbb{BD}_1/\hbar^2$$

given by sending the multiplication to $\frac{ab + (-1)^{|a||b|}ba}{2}$.

Given a \mathbb{BD}_1 -algebra A , we let A^{op} be the opposite algebra with the operations

$$\begin{aligned} a \cdot^{\text{op}} b &= (-1)^{|a||b|} b \cdot a \\ \{a, b\}^{\text{op}} &= -\{a, b\}. \end{aligned}$$

There are also lower-dimensional and higher-dimensional versions of the \mathbb{BD}_n operad.

Definition 5.2. A \mathbb{BD}_0 -algebra is a complex A over $k[[\hbar]]$ together with a degree 1 Lie bracket and a unital commutative multiplication satisfying the relations

- $d(ab) = d(a)b + (-1)^{|a|}ad(b) + \hbar\{a, b\},$
- $\{x, yz\} = \{x, y\}z + (-1)^{|y||z|}\{x, z\}y.$

In the classical limit we have an isomorphism

$$\mathbb{BD}_0/\hbar \cong \mathbb{P}_0$$

since then the multiplication is compatible with the differential. In the quantum case $\hbar \neq 0$ we have

$$\mathbb{BD}_0[\hbar^{-1}] \cong \widehat{\mathbb{E}}_0 \otimes k((\hbar)),$$

where the operad $\widehat{\mathbb{E}}_0$ is contractible, i.e. quasi-isomorphic to the operad \mathbb{E}_0 controlling complexes with a distinguished vector.

Next we are going to define the operad \mathbb{BD}_2 interpolating between \mathbb{P}_2 and the brace operad. Let us first recall that from a brace algebra A we can construct a Lie bracket of degree -1 given by

$$[x, y] = (-1)^{|x|}x\{y\} + (-1)^{|y|(|x|+1)}y\{x\}.$$

We will define the \mathbb{BD}_2 operad which captures the fact that the Lie bracket vanishes to the first order at $\hbar = 0$.

Definition 5.3. A \mathbb{BD}_2 -algebra is a dga A over $k[[\hbar]]$ together with a degree -1 Lie bracket denoted by $\{, \}$ and brace operations $A \otimes A^{\otimes n} \rightarrow A[-n]$ for $n > 0$ denoted by $x\{y_1, \dots, y_n\}$ satisfying the following relations:

- (Associativity).

$$x\{y_1, \dots, y_n\}\{z_1, \dots, z_m\} = \sum (-1)^\epsilon \hbar^q x\{z_1, \dots, z_{i_1}, y_1\{z_{i_1+1}, \dots\}, \dots, y_n\{z_{i_n+1}, \dots\}, \dots, z_m\},$$

where the sum goes over the locations of the y_i insertions and the length of each y_i brace. The sign is

$$\epsilon = \sum_{p=1}^n (|y_p| + 1) \sum_{q=1}^{i_p} (|z_q| + 1).$$

The exponent q is determined by assigning weight -1 to \hbar and weight 1 to braces $x\{y_1, \dots, y_n\}$ with $n \geq 2$ and forcing the equation to be homogeneous with respect to this grading.

- (Higher homotopies).

$$\begin{aligned} \hbar^{\delta_{n,2}} d(x\{y_1, \dots, y_n\}) &= \hbar^{\delta_{n,2}} (dx)\{y_1, \dots, y_n\} \\ &\quad + \hbar^{\delta_{n,2}} \sum_i (-1)^{|x| + \sum_{q=1}^{i-1} |y_q| + i} x\{y_1, \dots, dy_i, \dots, y_n\} \\ &\quad + \sum_i (-1)^{|x| + \sum_{q=1}^i |y_q| + i + 1} x\{y_1, \dots, y_i y_{i+1}, \dots, y_n\} \\ &\quad - (-1)^{(|y_1| + 1)|x|} y_1 \cdot x\{y_2, \dots, y_n\} \\ &\quad - (-1)^{|x| + \sum_{q=1}^{n-1} |y_q| + n} x\{y_1, \dots, y_{n-1}\} \cdot y_n, \end{aligned}$$

where $\delta_{n,2} = 1$ for $n = 2$ and zero otherwise.

- (Distributivity).

$$\sum_{k=0}^n (-1)^{|x_2|(\sum_{q=1}^k |y_q| + k)} \hbar^q x_1\{y_1, \dots, y_k\} x_2\{y_{k+1}, \dots, y_n\} = (x_1 \cdot x_2)\{y_1, \dots, y_n\},$$

where the exponent q is again determined by the weight grading.

- (Lie bracket).

$$\hbar\{x, y\} = (-1)^{|x|} x\{y\} + (-1)^{|y|(|x|+1)} y\{x\}.$$

- (Homotopy Leibniz rule).

$$\begin{aligned} d(x\{y_1, y_2\}) &= (dx)\{y_1, y_2\} - (-1)^{|x|+1} x\{dy_1, y_2\} - (-1)^{|x|+|y_1|} x\{y_1, dy_2\} \\ &= (-1)^{|y_1|} \{x, y_1 y_2\} - (-1)^{|y_1|} \{x, y_1\} y_2 - (-1)^{|x|+|y_1|} y_1 \{x, y_2\}. \end{aligned}$$

It is clear from the definition that

$$\mathbb{BD}_2[\hbar^{-1}] \cong \text{Br} \otimes k((\hbar)),$$

where Br is the operad controlling brace algebras. Let us also denote

$$\widetilde{\mathbb{P}}_2 = \mathbb{BD}_2/\hbar.$$

It is a weak version of the \mathbb{P}_2 operad as the next proposition shows. Before we state it, let us note that we have an obvious morphism

$$\widetilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$$

which sends all braces to zero.

Proposition 5.4. *The morphism $\widetilde{\mathbb{P}}_2 \rightarrow \mathbb{P}_2$ is a quasi-isomorphism of operads.*

Remark. Both $\widehat{\mathbb{P}}_2$ and $\widetilde{\mathbb{P}}_2$ are weakened versions of the \mathbb{P}_2 operad. While $\widehat{\mathbb{P}}_2$ corresponds to a weakening of the Jacobi identity in \mathbb{P}_2 while keeping the multiplication strictly associative and commutative, in the operad $\widetilde{\mathbb{P}}_2$ we weaken the Leibniz rule and the commutativity of the multiplication.

5.2. Modules. Let us now describe modules over \mathbb{BD}_1 -algebras.

Recall that a coisotropic morphism $A \rightarrow B$ for A a \mathbb{P}_1 -algebra is the data of a \mathbb{P}_0 -algebra on B and a morphism of \mathbb{P}_1 -algebras

$$A \rightarrow Z(B) \cong \widehat{\text{Sym}}(\text{T}_B).$$

For B a commutative graded algebra we denote by $\widehat{\text{D}}_\hbar(B)$ the completed algebra of \hbar -differential operators. That is, it is an algebra over $k[[\hbar]]$ generated by elements of B and T_B with the relations

$$\begin{aligned} vw - (-1)^{|v||w|}wv &= \hbar[v, w], \quad v, w \in \text{T}_B \\ vb - (-1)^{|v||b|}bv &= \hbar v.b, \quad v \in \text{T}_B, w \in B \end{aligned}$$

completed with respect to the increasing filtration given by the order of differential operators.

If B is a \mathbb{BD}_0 -algebra, the data of the differential on B determines a Maurer–Cartan element in $\widehat{\text{D}}_\hbar(B)$ and we denote by $Z(B)$, the \mathbb{BD}_0 -center of B , the algebra $\widehat{\text{D}}_\hbar(B)$ with the differential twisted by that Maurer–Cartan element. It is clear that $Z(B)$ is a \mathbb{BD}_1 -algebra.

More generally, if B is a commutative graded algebra the data of a Maurer–Cartan element in $\widehat{\text{D}}_\hbar(B)$ will be called a $\widehat{\mathbb{BD}}_0$ -algebra structure on B . Note that \mathbb{BD}_0 -structures correspond to those Maurer–Cartan elements which have order at most 2.

Let A be another \mathbb{BD}_1 -algebra.

Definition 5.5. A *left \mathbb{BD}_1 -module* over A is a \mathbb{BD}_0 -algebra B together with a morphism $A \rightarrow Z(B)$ of \mathbb{BD}_1 -algebras.

A *right \mathbb{BD}_1 -module* over A is the same as a left \mathbb{BD}_1 -module over A^{op} .

It is clear that the definition at $\hbar = 0$ reduces to the definition of a coisotropic morphism. Thus, one can talk about *quantizations* of a given coisotropic morphism $A_0 \rightarrow B_0$: these are \mathbb{BD}_1 -algebras A and \mathbb{BD}_0 -algebras B reducing to the given algebras A_0, B_0 at $\hbar = 0$ together with a left \mathbb{BD}_1 -module structure on B .

5.3. From \mathbb{BD}_1 to \mathbb{BD}_0 . We are now going to sketch a \mathbb{BD}_1 -version of Theorem 1.9.

Let A be a \mathbb{BD}_1 -algebra. In particular, it is a dga and so we have a dg coalgebra $T_\bullet(A[1])$. Introduce a commutative multiplication on $T_\bullet(A[1])$ given by the shuffle product and the Lie bracket given by (7).

Since A is not necessarily commutative, the differential is not compatible with the shuffle product. But its failure is exactly captured by the bracket.

Proposition 5.6. *Let A be a \mathbb{BD}_1 -algebra. The differential, multiplication and the bracket make $T_\bullet(A[1])$ into a \mathbb{BD}_0 -algebra compatibly with the coalgebra structure.*

We can also add modules in the picture. Let M be a left \mathbb{BD}_1 -module and N right \mathbb{BD}_1 -module over A . Then on the two-sided bar complex

$$N \otimes T_\bullet(A[1]) \otimes M$$

we can introduce the usual bar differential and the shuffle product.

Theorem 5.7. *Let A be a \mathbb{BD}_1 -algebra, M a left \mathbb{BD}_1 -module and N a right \mathbb{BD}_1 -module over A . Then the two-sided bar complex $N \otimes T_\bullet(A[1]) \otimes M$ has a $\widehat{\mathbb{BD}}_0$ -structure.*

At $\hbar = 0$ this construction recovers the $\widehat{\mathbb{P}}_0$ -structure of Theorem 1.9.

5.4. From \mathbb{BD}_2 to \mathbb{BD}_1 . We finish with a similar construction of a \mathbb{BD}_1 -structure on the bar complex of a \mathbb{BD}_2 -algebra.

Let A be a \mathbb{BD}_2 -algebra. Using the dga structure on A one endows the bar complex $T_\bullet(A[1])$ with the structure of a dg coalgebra. We introduce the multiplication by

$$\begin{aligned} & [x_1 | \dots | x_n] \cdot [y_1 | \dots | y_m] \\ &= \sum_{\{i_p, l_p\}_{p=1}^n} (-1)^\epsilon \hbar^q [y_1 | \dots | y_{i_1} | x_1 \{y_{i_1+1}, \dots, y_{i_1+l_1}\} | \dots | x_n \{y_{i_n+1}, \dots, y_{i_n+l_n}\} | \dots | y_m], \end{aligned}$$

where the sign is

$$\epsilon = \sum_{p=1}^n (|x_p| + 1) \sum_{q=1}^{i_p} (|y_q| + 1)$$

and the exponent q of \hbar is given by the number of brace operations with $l_i > 1$.

It is not difficult to see that the commutator $a \cdot b - (-1)^{|a||b|} b \cdot a$ of any two elements $a, b \in T_\bullet(A[1])$ vanishes at $\hbar = 0$. Therefore, one can define the Lie bracket by

$$\{a, b\} = \frac{ab - (-1)^{|a||b|} ba}{\hbar}.$$

Proposition 5.8. *Let A be a \mathbb{BD}_2 -algebra. Then the bar complex $T_\bullet(A[1])$ is a \mathbb{BD}_1 -algebra compatibly with the coalgebra structure.*

This proposition interpolates between the \mathbb{P}_1 -structure of Proposition 1.6 and the dga structure of Proposition 3.3.

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